MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2003

Round 1 Arithmetic and Number Theory

1. ________________________

2. ________________________

3. ________________________

1. How many 2-digit prime numbers have the property that the sum of their digits is also a 2-digit prime number?

2. A survey of the 810 graduates of Parallel High School showed that 582 of them had taken at least one foreign language. 209 took French, 244 took German, and 338 took Spanish. Of these numbers 25 took all three, of students that took exactly two languages, 53 took German and Spanish, and 14 more took French and Spanish than took French and German. How many students took both French and Spanish but not German?

3. Let A and B be positive digits which are not necessarily distinct. Determine the number of distinct ordered pairs (A, B) such that $.\overline{A} + .\overline{B} = .\overline{C} < .99$

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Round 2 Algebra 1

1. ________________________

2. ________________________

3. ________________________

1. A collection of nickels and dimes is worth \$2.35. If twice the number of dimes is three more than three times the number of nickels, how many dimes are there?

2. If $AB_7 = BA_5$, with A and B non-zero, determine the ordered pair (A, B)

3. Let A be a two digit number with distinct digits. Let B be the number formed by reversing the digits of A, and let k be an integer. If $A^2 - B^2 = k^2$, what is the value of A?

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Round 3 – Geometry

1. ______________________

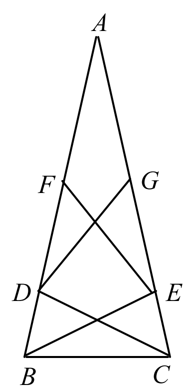
2. ______________________

3. ______________________

1. In ΔABC, AB = 6, BC = 8, and AC = 10. $\overline{BD}$ is an altitude and point E is such that BECD is a rectangle. How many square units are in the area of the rectangle?

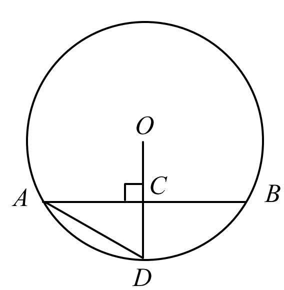
2. ΔABC is isosceles and $AF = AG = FE = GD = DC = EB = BC$.

   Find $m\angle A$ in degrees.



3. O is the center of the circle shown.

   $OC = 3$ and $AD = 36$. Find CD.



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STATE PLAYOFFS – 2003

Round 4 – Algebra 2

1. ________________________

2. ________________________

3. ________________________

1. The sum of the reciprocal of a positive number and the square of its reciprocal is $\frac{40}{9}$. What is the number?

2. Determine the solution of $5|x| \le 2|x+1|$. Express your answer using inequality notation.

3. Starting with $q_1 = 2x^2 - 6x + 3$, a sequence of quadratics is formed such that the zeros of q_n are twice the zeros of q_{n-1} for $n \ge 2$. If $q_7 = x^2 - bx + c$, find the ordered pair (b, c).

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STATE PLAYOFFS – 2003

Round 5 – Analytic Geometry

1. ________________________

2. ________________________

3. ________________________

1. Given A(-2, 5), B(2, -7) and C(3,8). If point M lies on $\overline{AB}$ such that AM = $\frac{3}{4}$ AB, the equation of the line through M and C can be written as ax + by = c, where a, b, and c are integers with no common factor and a > 0. Write the ordered triple (a, b, c).

2. The intersection points of $6x^2 + y^2 = 49$ and $y = x^2 - 9$ can be connected to form a trapezoid. In exact form, find the number of square units in the area of this trapezoid.

3. Determine all values of k, $k \neq 0$, such that the following system has exactly three solutions:

   $x^2 + (y - 1)^2 = 1$ and $y = kx^2$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2003

Round 6 – Trig and Complex Numbers

1. ________________________

2. ________________________

3. ________________________

1. Determine, in exact form, the value of $\sin\left(2\text{Cos}^{-1}\frac{3}{8}\right)$.

2. The combined solutions of the equations $x^2 - 4x + 8 = 0$ and $x^2 + 2x + 10 = 0$ are 4 complex numbers, which if graphed, will appear one per quadrant. The product of the solutions in the first and third quadrants can be written as a + bi. Write that product as the ordered pair (a, b).

3. There is a real number x such that the sides of a triangle have lengths $\sin x$, $\cos x$, and $\tan x$ and the product of the sides is 1/4. Determine the cosine of the largest angle.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2003

Team Round

1. ________________________ **4.** ____________________

2. _________________________ **5.** ____________________

3. ________________________ **6.** ____________________

1. Trapezoid $ABCD$ has bases $\overline{AB}$ and $\overline{CD}$ with $AB < CD$. Let $\overline{MN}$ be the median. If $AB = \log_a b$ and $CD = \log_{b^2} a$, determine the greatest lower bound for MN.

2. A circle with center in the first quadrant is tangent to $y = x + 10$, $y = x - 6$, and the y–axis. Let (h, k) be the center of the circle. Determine the sum $h + k$.

3. Let R be the region in the first quadrant bounded by the x– and y–axis and the graphs of $f(x) = \frac{9}{25}x + b$ and $f^{-1}(x)$. If the area of R is 49, determine the value of b.

4. Determine all values of p such that there is no real solution to the following system:

$$x^2 + y^2 = 32 \text{ and } (x - p)^2 + (y - p)^2 = 128.$$

5. Two complex numbers $a + bi$ and $c + di$ are <u>almost conjugates</u> if and only if $|a - c| \leq 1$ and $|b + d| \leq 1$. Determine the <u>almost conjugate</u> of $2 + 3i$ that is furthest from the origin.

6. Determine the sum of all integer palindromes between 10,000 and 99,999 that are divisible by 25.

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Answer Sheet

Round 1

1. 5
2. 60
3. 28

Round 2

1. 18
2. (2, 3)
3. 65

Round 3

1. $\frac{768}{25}$ or 30.72
2. $\frac{180}{7}$
3. 24

Round 4

1. $\frac{3}{5}$
2. $-\frac{2}{7} \le x \le \frac{2}{3}$
3. (192, 6144)

Round 5

1. $(6, -1, 10)$
2. $8+8\sqrt{2}$
3. $k > \frac{1}{2}$

Round 6

1. $\frac{3\sqrt{55}}{32}$
2. $(4, -8)$
3. $-\frac{\sqrt{3}}{6}$

Team

1. $\frac{\sqrt{2}}{2}$
2. $2+8\sqrt{2}$
3. 28/5
4. $p < -12$ or -4<p<4 or p>12
5. $3-4i$
6. 1,100,000

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2003 – Solutions Outline

Round 1 Arithmetic and Number Theory

1. 29, 47, 67, 83, 89

2. Let x = # taking F & G, f, only French, g, only German, and s, only Spanish. Now f + 2x + 39 = 209, g + x + 78 = 244, s + x + 92 = 338, and f + g + s + 2x + 92 = 582. Subtracting the first from the last gives g + s = 320 and subtracting the second from the third gives s – g = 80.

3. We have $\frac{A}{9} + \frac{B}{9} = \frac{C}{9}$, giving $A + B = C$ for 2 $C < 9$. For $A = 1, 2, 3, 4$, $B = 1, 2, 3, 4$ giving $4 \cdot 4 = 16$ ordered pairs. For $A = 5$, $B = 1, 2, 3$; for $A = 6$, $B = 1, 2$; and for $A = 7$, $B = 1$. Since each of these pairings can be reversed, i.e. (5, 1) also gives (1, 5), we have a total of 16 + 6 + 4 + 2 = 28 ordered pairs.

Round 2 Algebra 1

1. 5n + 10d = 235, 2d = 3n + 3

2. $7a + b = 5b + a \to 6a = 4b \to a = \frac{2}{3}b$, only possibility is $b = 3$

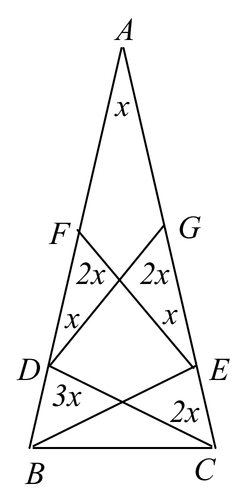
3. Let $A = 10a + b$ and $B = 10b + a$. Then $(10a + b)^2 - (10b + a)^2 = 99a^2 - 99b^2 = 9 \cdot 11\left(a^2 - b^2\right)$.

   For the left side to be a perfect square, $a^2 - b^2$ must equal $11m^2$. If $m = 1$, then $a = 6$ and $b = 5$. If m 2, there are no digits satisfying the equation. Thus, $A = 65$.

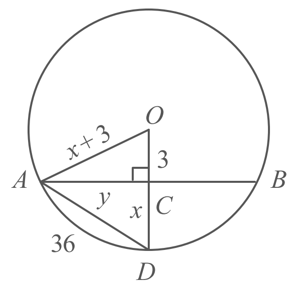
Round 3 – Geometry

1. $BD = \frac{24}{5}$; $DC = \frac{32}{5}$

2. Let $m\angle A = x$. Since $AF = FE$, $m\angle FEA = x$. By the Exterior Angle Theorem, $m\angle DFE = x + x = 2x$. Similarly, $m\angle DGC = 2x$. Since $DG = DC$, then $m\angle DCG = 2x$. Thus, $m\angle GDC = 180 - 4x$, making $m\angle BDC = 3x$. Since $DC = BC$, $m\angle DBC = 3x$.. Since ΔABC is isosceles, $m\angle ECB = 3x$ and $m\angle A + m\angle ABC + m\angle ACB = x + 3x + 3x = 7x$. Thus, $m\angle A = \frac{180}{7}$.



3. $x^2 + y^2 = 36^2$ and $y^2 + 3^2 = AO^2 = OD^2 = (x + 3)^2$.
Substituting $(x + 3)^2 - 3^2$ for y^2 in the first equation gives
$x^2 + 3x - 648 = 0 \rightarrow (x + 27)(x - 24) = 0$.
Thus, $x = 24$.



Round 4 – Algebra 2

1. $\frac{1}{x}+\left(\frac{1}{x}\right)^2=\frac{40}{9} \rightarrow 40x^2 - 9x - 9 = 0 \rightarrow (5x - 3)(8x + 3) = 0$

2. For $x > 0$, $5x \le 2x + 2 \rightarrow x \le 2/3$: For $-1 < x < 0$, $-5x \le 2x + 2 \rightarrow x \ge -2/7$. For $x \le -1$, no solution.

3. Comparing $(x - m)(x - n) = x^2 - (m + n)x + mn$ with $(x - 2m)(x - 2n) = x^2 - 2(m + n) + 4mn$ we see that each time we move to the next quadratic in the sequence the sum of the roots is doubled while the product is multiplied by 4. Thus, in q_7 we have $b = 3 \cdot 2^6 = 192$ and $c = \frac{3}{2} \cdot 4^6 = 6144$.
Answer (192, 6144).

Round 5 – Analytic Geometry

1. M(1, –4); $m\overline{MC} = 6$; $y - 8 = 6(x - 3)$; $y = 6x - 10$ (6, –1, 10)

2. By substitution, get $6x^2 + x^4 - 18x^2 + 81 = 49 \rightarrow x^4 - 12x^2 + 32 = 0 \rightarrow x = \pm 2$ or $x = \pm 2\sqrt{2}$.
The points of intersection are (2, -5), (-2, -5), $(2\sqrt{2}, -1)$, $(-2\sqrt{2}, -1)$. The bases of the trapezoid have lengths 4 and $4\sqrt{2}$, and the height is 4.

3. Substituting kx^2 for y we obtain $k^2x^4 + (1 - 2k)x^2 = 0$. Since $x = 0$ is obviously a solution, divide by x^2 to obtain $k^2x^2 + (1 - 2k) = 0$. This has two solutions if the discriminant is greater than 0. So, $0^2 - 4k^2(1 - 2k) > 0$. This is true if $1 - 2k$ is negative, so $1 - 2k < 0 \rightarrow k > 1/2$.

Round 6 – Trig and Complex Numbers

1. $\sin\left(2\cos^{-1}\frac{3}{8}\right) = 2\sin\left(\cos^{-1}\frac{3}{8}\right)\cos\left(\cos^{-1}\frac{3}{8}\right) = 2\frac{\sqrt{55}}{8}\frac{3}{8}$

2. Using the quadratic formula, the solutions are $2 \pm 2i$ and $-1 \pm 3i$. $(2 + 2i)(-1 - 3i) = 4 - 8i$

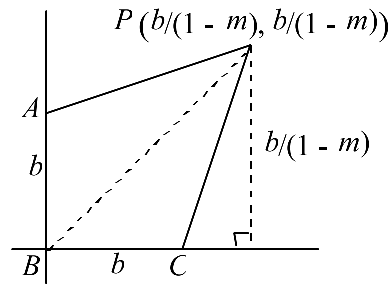
3. Since $(\sin x)(\cos x)(\tan x) = \frac{1}{4}$, then $(\sin x)(\cos x)\frac{\sin x}{\cos x} = \frac{1}{4} \rightarrow \sin^2 x = \frac{1}{4} \rightarrow \sin x = \frac{1}{2}$. Thus, $x = 30°$ and since $\tan 30° = \frac{\sqrt{3}}{3} = .577\ldots$ and $\cos 30° = \frac{\sqrt{3}}{2} = .866\ldots$, the longest side is $\cos 30°$. Let θ be the angle opposite $\cos 30°$. Then $\left(\frac{\sqrt{3}}{2}\right)^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 - 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{3}\right)\cos\theta \rightarrow \cos\theta = \frac{-\sqrt{3}}{6}$.

Team Round

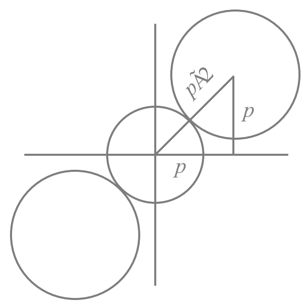
1. $MN = \dfrac{\log_a b + \log_{b^2} a}{2} = \dfrac{\log_a b + \dfrac{1}{\log_a b^2}}{2} = \dfrac{\log_a b + \dfrac{1}{2\log_a b}}{2}$. Using the arithmetic mean-geometric mean inequality, $MN \quad \sqrt{\log_a b \cdot \dfrac{1}{2\log_a b}} = \dfrac{1}{\sqrt{2}}$. Equality occurs when $\log_a b = \log_{b^2} a$ which can't be since if $AB = DC$ then $ABCD$ is not a trapezoid. So while MN can't equal $\frac{1}{\sqrt{2}}$, $MN > \frac{1}{\sqrt{2}}$, so $\frac{\sqrt{2}}{2}$ is the greatest lower bound for MN.

2. Using $x - y + 10 = 0$, $x - y - 6 = 0$, and center $C(h, k)$, then C is equidistant from the lines, giving $\dfrac{|h - k + 10|}{\sqrt{2}} = \dfrac{|h - k - 6|}{\sqrt{2}}$ using the formula for the distance from a point to a line. Using $h - k + 10 = -(h - k - 6)$ we obtain $k = h + 2$. Since the circle is tangent to the y–axis, the radius of the circle is h. Thus, $\dfrac{|h - k + 10|}{\sqrt{2}} = h$ and letting $k = h + 2$ we obtain $\dfrac{|h - (h + 2) + 10|}{\sqrt{2}} = h$ $\rightarrow \dfrac{8}{\sqrt{2}} = h$. Thus, $h = 4\sqrt{2}$, $k = 4\sqrt{2} + 2$ and their sum is $2 + 8\sqrt{2}$. Note: had we set $h - k + 10 = h - k - 6$ or $-(h - k + 10) = -(h - k - 6)$, we would have a negative answer for k.

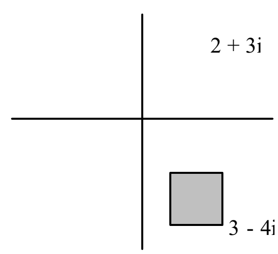
3. If $f(x) = mx + b$, then $f^{-1}(x) = \dfrac{x - b}{m}$ and their point of intersection can be found by setting $x = mx + b$ since they intersect on $y = x$. Thus $x = \dfrac{b}{1 - m}$ and the point of intersection is $\left(\dfrac{b}{1-m}, \dfrac{b}{1-m}\right)$. Region R can be broken up into congruent triangles PAB and PCB which both have a base of b and a height of $\dfrac{b}{1 - m}$. The area of R is $2\left(\dfrac{1}{2}b\right)\left(\dfrac{b}{1-m}\right) = \dfrac{b^2}{1 - m} = 49$. For $m = \dfrac{9}{25}, b^2 = \dfrac{16}{25} \cdot 49 \rightarrow b = \dfrac{28}{5}$.



4. As shown in the diagram, when the circles are tangent, the distance between the centers is $|p|\sqrt{2}$. Thus, $|p|\sqrt{2} = 4\sqrt{2} + 8\sqrt{2} \rightarrow |p| = 12$ The system will have no real solution when the circles do not intersect. That occurs for $p > 12$ or $p < -12$ or $-4 < p < 4$.



5. The set of points for which $|2 - c| \le 1$ and $|3 + d| \le 1$ is the square defined by $1 \le c \le 3$ and $-2 \le d \le -4$. As shown by the diagram the point furthest from the origin is $3 - 4i$.



6. The palindromes can't end in 00 or 50, so they end in either 25 or 75. This means that the palindromes are of the form 52X25 or 57X75 where X can represent any digit from 0 to 9. Thus, there are 10 numbers of the form 52X25 and 10 of the form 57X75. Hence the sum of the 20 palindromes is 20(50,000) + 10(2000) + 10(7000) + 10(20) + 10(70) + 20(5) + 100(2)(0 + 1 + 2 + . . . + 9). This equals 1,000,000 + 10(9090) + 100 + 200(45) = 1,100,000.