MAML

STATE INVITATIONAL MATH LEAGUE COMPETITION

April 9, 2010

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2010

Round 1 **Arithmetic and Number Theory**

1. ______________________

2. ______________________

3. ______________________

1. What is the digit in the 2010th place of the product of $(.\overline{63})(.\overline{571428})$?

2. Given a six-digit number $N = ABC,ABC$ with A, B, and C not necessarily distinct and $A \neq 0$, find the <u>smallest</u> N such that N is the product of 5 distinct primes.

3. Let $R = .\overline{AB}$. If R is written as the ratio of two integers in simplest form, for how many values of R is the denominator 99?

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2010

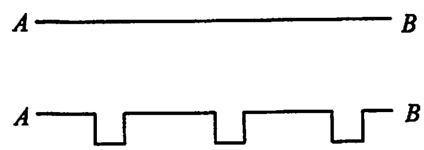
Round 2 **Algebra 1**

1. ____________________

2. ____________________

3. ____________________

1. Line segment $\overline{AB}$ has a length of 20. Keeping the distance from A to B fixed at 20, a number of congruent crenellations are added. Their depth is an integer and is the same length as their width. The diagram illustrates the case of three crenellations. If the length of the path equals 50, find the largest number of crenellations.



2. Let x and y be integers with $1 \le x, y \le 2010$.
Determine the number of ordered pairs (x, y) such that $2^{x-y} + 2^{y-x+3} = 6$.

3. Given: integers A, B, C, and D such that $AB_{12} + CD_{12} = ABC_{12}$
Determine the base 10 sum $A + B + C + D$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2010

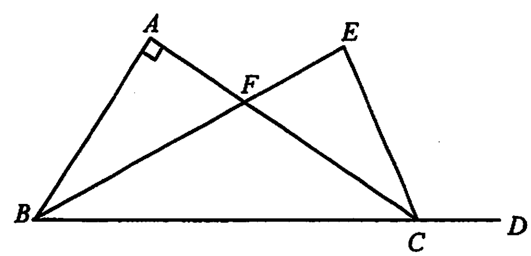
Round 3 – Geometry

1. ____________________

2. ____________________

3. ____________________

1. $\overline{BE}$ bisects $\angle ABC$, $\overline{AC}$ bisects $\angle ECB$, and $\overline{BA} \perp \overline{CA}$.
   $m\angle E + m\angle ECD = k(m\angle ABF)$.
   Determine k.



2. Two sides of a regular 11-sided figure are extended to form an angle P. Determine the smallest possible measure for $\angle P$.

3. Let $a, b,$ and c be <u>distinct</u> integers. The shorter two sides of a triangle have lengths $\sqrt{2a}$ and $\sqrt{3b}$; the length of the longest side is $\sqrt{c}$. Each pair of values of a and b determines a set of values of c and in each such set there is a largest value. Determine the <u>minimum</u> of the largest values of c.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2010

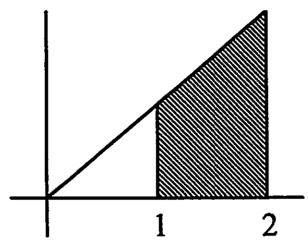
Round 4 – Algebra 2

1. ______________________

2. (________ , ________)

3. ______________________

1. Shown is a graph of $y = (\log k)x$. If the area of the trapezoidal shaded region under the graph equals $\log(216)$, determine the value of *k*.



2. Ralph devised a scale for measuring the hotness of peppers. First, he measured the amount A of capsaicin in a cubic centimeter of the pepper. Then he calculated one hundred times the reciprocal of the log of A. He used a base of 10. Finally, he took the negative of that number to obtain the final result. A jalapeno is a 40 on the scale and a cayenne is a 60. If the exact ratio of the amount of capsaicin in a cubic centimeter of cayenne pepper to the amount in a jalapeno is $10^{m/n}$, where *m* and *n* are relatively prime, find the ordered pair (m, n).

3. The first three terms of an arithmetic progression are $x + 2, |x + 4|, |x| + 7$. If the 4th term is $x + k$, find all values for *k*.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2010

Round 5 – Analytic Geometry

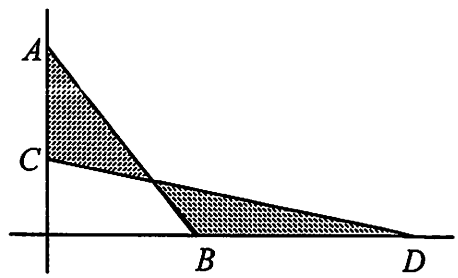
1. ____________________

2. ____________________

3. ____________________

1. Determine the radius of the largest circle that is internally tangent to both circles whose equations are $(x-2)^2 + y^2 = 4$ and $x^2 + (y-2)^2 = 4$.

2. If the equation of $\overline{AB}$ is $y = mx + b$ and the equation of $\overline{CD}$ is $y = \frac{x}{m} + \frac{1}{b}$ and the areas of the shaded regions are equal, find m in terms of b.



3. Vertices A and B of trapezoid $ABCD$ lie on the positive x-axis and vertices C and D lie on the graph of $y = x^2$ with $\overline{DA}$ and $\overline{CB}$ perpendicular to the x-axis. If $AB = 1$ and the slope of $\overline{CD}$ numerically equals the area of $ABCD$, determine the number of units in the area of $ABCD$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2010

Round 6 – Trig and Complex Numbers

1. ______________________

2. __________ **:** __________

3. (______ **,** ______ **,** ______ **)**

1. Let $z = a + bi$ be the set of complex numbers such that $(1 - 2i)z = r$ where r is a real number. If the points (a, b) are plotted in the xy-coordinate plane, they lie on the graph of the function $y = f(x)$. Express b in terms of a.

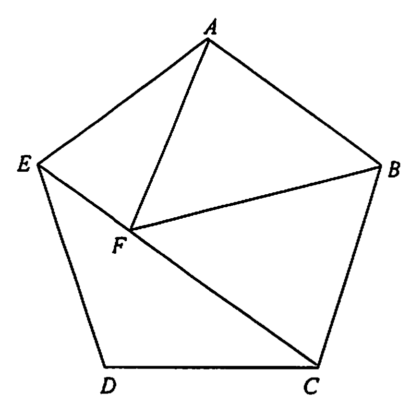
2. In ΔABC, $m\angle A = 60°, a = 7,$ and $c = 8$. Determine the ratio of the areas of the triangles determined by the data, larger over smaller.

3. $ABCDE$ is a regular pentagon of side 2. If

   $AE = AF$, then

   $FB^2 - FE^2 = M\cos^2 36° + N\cos 36° + K$.

   Determine the ordered triple (M, N, K).



MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2010

Team Round

1. ____________________ **4.** ____________________

2. ____________________ **5.** (________ , ________)

3. ____________________ **6.** ____________________

1. In trying to solve $\log_4(x-a)+\log_4(x-b)=1$, Goofus wrote $\log_4\dfrac{x-a}{x-b}=1$, solved that equation correctly and obtained the correct answer to the original problem. For $a,b\in\{1,2,3,\ldots,2009\}$ with $a<b$, how many ordered pairs (a,b) would allow for both equations to have the same solution?

2. Let $f(x)=mx+b$ for $m\neq 0$ and $b>4$. If $f(f(f(2)))=f(2)$, determine the largest possible value of m.

3. Point $P(x,y)$ moves so that the sum of its distances from the circles $(x+1)^2+y^2=1$ and $(x-4)^2+y^2=4$ is 7. Find the largest possible y-coordinate that P can achieve.

4. For $0 \le x \le 2\pi$, determine the number of solutions to $\frac{\sin x \cdot \sin 2x \cdot \sin 4x}{\cos x} = 0$.

5. Start with a large rectangular piece of paper. If a strip $\frac{1}{2}$ inch wide is cut off all around the piece of paper, then the area of the rectangle is reduced by two square feet. The maximum possible area in square feet of the original piece of paper can be written as $\left(\frac{a}{b}\right)^2$ for positive integers a and b. Determine the ordered pair (a, b).

6. Let x and y be positive integers such that $x + y + xy = N$ for $N = \{1, 2, 3, \ldots, 50\}$. Determine the number of values of N for which the sum $x + y$ is unique.

4. For $0 \le x \le 2\pi$, determine the number of solutions to $\dfrac{\sin x \cdot \sin 2x \cdot \sin 4x}{\cos x} = 0$.

5. Start with a large rectangular piece of paper. If a strip $\frac{1}{2}$ inch wide is cut off all around the piece of paper, then the area of the rectangle is reduced by two square feet. The maximum possible area in square feet of the original piece of paper can be written as $\left(\frac{a}{b}\right)^2$ for positive integers a and b. Determine the ordered pair (a, b).

6. Let x and y be positive integers such that $x + y + xy = N$ for $N = \{1, 2, 3, \ldots, 50\}$. Determine the number of values of N for which the sum $x + y$ is unique.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2010

Answer Sheet

Round 1

1. 6
2. 106,106
3. 60

Round 2

1. 15
2. 4017
3. 23

Round 3

1. 7
2. $\frac{180}{11}$
3. 13

Round 4

1. 36
2. $(5,6)$
3. 35

Round 5

1. $2-\sqrt{2}$
2. $-b^2$
3. $2+\sqrt{3}$

Round 6

1. $b=\frac{2}{3}a$ b=2a
2. $\frac{5}{3}$ or 5:3
3. $(-16,8,8)$

Team

1. 2006
2. -1
3. $\frac{5\sqrt{3}}{2}$
4. 7
5. $(289, 24)$
6. 19

m p=9d

Solutions: State Meet 2010

Round 1 Arithmetic and Number Theory:

1. The product equals $\frac{7}{11}\cdot\frac{4}{7}=\frac{4}{11}=.\overline{36}$. The digit in the 2010th place is $\boxed{6}$.

2. $ABC,ABC = 1001(ABC) = 7\cdot 11\cdot 13(ABC)$. We seek the smallest number ABC that is the product of two distinct primes. That would be $(2)(53) = 106$ since $(3)(37) = 111$ and $(5)(23) = 115$. Thus $ABC = 106$ and $N = (1001)(106) = \boxed{106{,}106}$.

3. $N = \frac{10A+B}{99} = \frac{10A+B}{3\cdot 3\cdot 11}$. We need to determine the number of numbers from 1 to 99 that don't have a 3 or an 11 as a factor. There are $\frac{99}{3}+\frac{99}{11}-\frac{99}{33} = 33+9-3 = 39$ that do, so there are $\boxed{60}$ that don't. Alternately, if ϕ denotes Euler's totient function, then the answer we seek is $\phi(99) = 99\left(1-\frac{1}{3}\right)\left(1-\frac{1}{11}\right) = 60$.

Round 2 Algebra I:

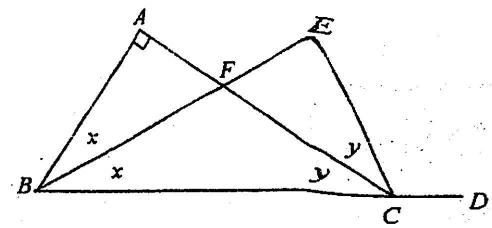
1. Let the number of crenellations be n and the depth and width of each be d. Since 30 was added to the length of the path and the addition consists of $2d$ for each crenellation, then $2nd = 30$ making $nd = 15$. The maximum occurs when $d = 1$ making $\boxed{n = 15}$.

2. Let $x-y=n$. Then $2^n + 2^{-n+3} = 6 \to 2^n + 2^3\cdot 2^{-n} = 6$.
$2^n + 8\cdot 2^{-n} = 6 \Rightarrow 2^n + \frac{8}{2^n} = 6 \to 2^{2n} - 6\cdot 2^n + 8 = 0 \to \left(2^n-2\right)\left(2^n-4\right) = 0$. Thus, $x-y=1$ or $x-y=2$. The former generates 2009 ordered pairs from (2, 1) to (2010, 2009) and the latter generates 2008 ordered pairs from (3, 1) to (2010, 2008).
Ans: $\boxed{4017}$.

3. $12A + B + 12C + D = 144A + 12B + C \to D = 132A + 11B - 11C$. If $A = 2$, then $D = 264 + 11(B-C)$, but the difference $B-C$ isn't enough to obtain a digit, so $A = 1$, giving $D = 132 + 11(B-C)$. If $B - C = -11$, then $D = 11$. For any greater value of $B-C$ such as -10, D will not be a digit. So there is one such number ABC_{12} and it is $1\,0\,(11)$. Answer: 23.

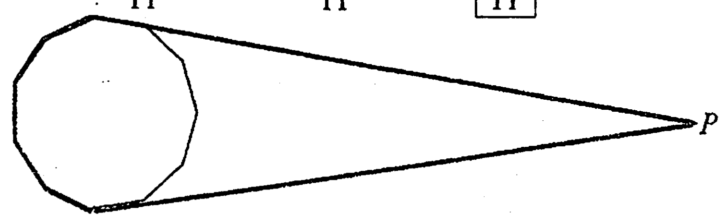
Round 3 Geometry:

1. Since $2x + y = 90 \rightarrow y = 90 - 2x$.
$m\angle E + m\angle ECD = (180 - (x + 2y)) + (180 - 2y)$. This gives
$m\angle E + m\angle ECD = 360 - x - 4y = 360 - x - 4(90 - 2x) =$
$7x$. Thus, $\boxed{k = 7}$.



2. Shown below is the 11-sided figure with angle P. The shaded lines form a 7-sided convex polygon, 6 of whose angles are angles of the 11-sided figure. Each angle measures $\frac{9 \cdot 180}{11}$. Thus,

$$m\angle P = (7 - 2)180 - 6 \cdot \frac{9 \cdot 180}{11} = \frac{(55 - 54)180}{11} = \boxed{\frac{180}{11}}$$



3. The smallest large value of c ought to occur with the smallest possible choices for a and b. There are two cases. If $a = 1$ and $b = 2$ we have $\sqrt{2} + \sqrt{6} > \sqrt{c} \rightarrow 8 + 4\sqrt{3} > c$. This gives approximately $14.928 > c$. If $a = 2$ and $b = 1$, we have $\sqrt{4} + \sqrt{3} > \sqrt{c} \rightarrow 7 + 4\sqrt{3} > c$. This gives approximately $13.928 > c$ so the smallest largest value of c is $\boxed{13}$.

Round 4 Algebra II:

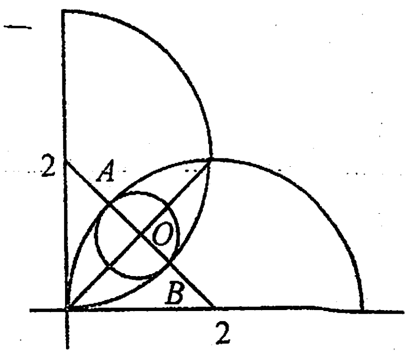
1. $\frac{1}{2} \cdot 2 \cdot 2(\ln k) - \frac{1}{2} \cdot 1 \cdot 1(\ln k) = \ln(216) \rightarrow \frac{3}{2}\ln k = \ln(216) \rightarrow \ln k = \ln(216)^{2/3} = \ln 36$. Thus, $k = \boxed{36}$.

2. $40 = -\frac{100}{\log J} \rightarrow \log J = -\frac{5}{2} \rightarrow J = 10^{-5/2}$ and $60 = -\frac{100}{\log C} \rightarrow \log C = -\frac{5}{3} \rightarrow C = 10^{-5/3}$. Thus, $\frac{C}{J} = \frac{10^{-5/3}}{10^{-5/2}} = \boxed{10^{5/6}}$.

3. Three cases: i) if $x \geq 0$, we have $x+2, x+4,$ and $x+7$. There is no common difference so no arithmetic progression. ii) if $-4 < x < 0$, we have $x+2, x+4,$ and $-x+7$. We solve $(x+4)-(x+2)=(7-x)-(x+4)$ and obtain $x=\frac{1}{2}$ but that lies outside the restricted domain so it is rejected. iii) if $x \leq -4$, we have $x+2, -x-4,$ and $-x+7$. Thus, $(-x-4)-(x+2)=(7-x)-(-x-4) \rightarrow -2x-6=11 \rightarrow x=-\frac{17}{2}$. This gives the sequence $-\frac{13}{2}, \frac{9}{2}, \frac{31}{2}$. The common difference is 11, so $x+k=\frac{53}{2} \rightarrow k=\frac{17}{2}+\frac{53}{2}$, so $\boxed{k=35}$.

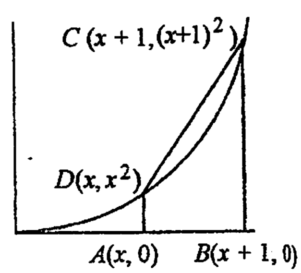
Round 5 __Analytic Geometry:__

1. The center of O lies on the line $y = x$. The length of the line connecting the centers is $2\sqrt{2}$ and since each radius is 2, the overlap of the two radii equals the diameter of the circle so the diameter equals $4-2\sqrt{2}$. The radius is $\boxed{2-\sqrt{2}}$.



2. Since $A=(0,b)$ and $B=\left(-\frac{b}{m},0\right)$, then $a(\Delta AOB)=-\frac{b^2}{2m}$. Since $C=\left(0,\frac{1}{b}\right)$ and $D=\left(-\frac{m}{b},0\right)$, then $a(\Delta COD)=-\frac{m}{2b^2}$. Since each triangle consists of a shaded region plus a common area, for the shaded regions to be equal, the areas of the triangles must be equal. Thus, $-\frac{b^2}{2m}=-\frac{m}{2b^2} \rightarrow m^2=b^4$. Since m is negative, $\boxed{m=-b^2}$.

3. Let $A=(x,0), B=(x+1,0), C=\left((x+1),(x+1)^2\right)$, and $D=(x,x^2)$. The slope of $\overline{CD}=2x+1$ and the area of $ABCD=\frac{1}{2}\cdot 1\left(x^2+2x+1+x^2\right)$. Thus, $4x+2=2x^2+2x+1 \rightarrow 2x^2-2x-1=0$ and $x=\frac{1+\sqrt{3}}{2}$. The area equals $2\left(\frac{1+\sqrt{3}}{2}\right)+1=\boxed{2+\sqrt{3}}$.



Round 6 Trigonometry and Complex Numbers:

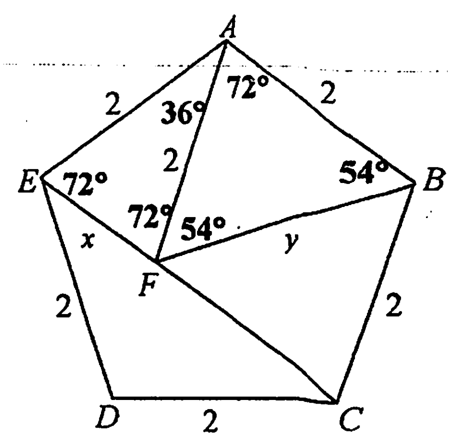
1. $(1-2i)z = r \rightarrow z = \dfrac{r}{1-2i} \rightarrow z = r\left(\dfrac{1}{5}+\dfrac{2}{5}i\right)$. Thus, the points on the function are of the form $\left(\dfrac{r}{5},\dfrac{2r}{5}\right)$. We recognize this as a parametric form of a line, i.e., $x=\dfrac{r}{5}, y=\dfrac{2r}{5}$, so $y = 2x$ is the answer. Intuitively, we could have reasoned that since the arguments of a complex number and its conjugate are θ and $-\theta$, their product will always result in a real number. Thus z must be the set of multiples of the conjugate of $1-2i$ and that set consists of numbers of the form $k(1+2i)$ where k is any real number. This set lies on the line $\boxed{y=2x}$. $\therefore\ b=\dfrac{2}{3}a$

   Alternate solution: $(-3-4i)(a+bi) = -1-18i \rightarrow -3a+4b=-1$ and $-4a-3b=-18$

2. Let x = the other side. Using the Law of Cosines $7^2 = x^2+8^2-2\cdot 8\cdot x\cdot\cos 60°$, giving $x^2-8x+15=0 \Rightarrow x=3$ or 5. Since the triangles have the same height, the ratio of their areas is the ratio of their bases so the answer is $\boxed{\dfrac{5}{3} \text{ or } 5:3}$.

   Alternate solution: Let the altitude from B intersect $\overline{AC}$ in D. Then $BD = 4\sqrt{3}$, $AD = 4$ and $BC = 1$ by the Pythagorean Theorem. Then $\dfrac{4+1}{4-1}=\dfrac{5}{3}$

3. Using the angles as marked, let $EF = x$ and $FB = y$. By the Law of Cosines, $x^2 = 2^2+2^2-2\cdot 2\cdot 2\cos 36$ and $y^2 = 2^2+2^2-2\cdot 2\cdot 2\cos 72$. Thus, $y^2-x^2 = 8(\cos 36-\cos 72)$. Since $\cos 72 = 2\cos^2 36 - 1$, $y^2-x^2 = 8\left(\cos 36° - (2\cos^2 36° - 1)\right) =$ $-16\cos^2 36 + 8\cos 36 + 8$. So $(M, N, K) = \boxed{(-16, 8, 8)}$.



Team:

1. From $\log_4 \frac{x-a}{x-b} = 1$ we obtain $x - a = 4(x-b) \to x = \frac{4b-a}{3}$. From $\log_4(x-a) + \log_4(x-b) = 1$ we obtain $(x-a)(x-b) = 4$. Using $x = \frac{4b-a}{3}$ we obtain $\left(\frac{4b-a}{3} - a\right)\left(\frac{4b-a}{3} - b\right) = 4 \to (4b-4a)(b-a) = 36 \to (b-a)^2 = 9$. Thus, since $b > a$ we have $b - a = 3$ giving ordered pairs (1, 4), (2, 5), . . . (2006, 2009). There are $\boxed{2006}$ ordered pairs that allow the mistake to yield the correct answer.

2. Clearly $f(f(2)) = 2$ so $f(f(2)) = f(2m+b) = 2m^2 + mb + b = 2$. We obtain $2m^2 + mb + (b-2) = 0 \to m = \frac{-b \pm \sqrt{b^2 - 4\cdot 2(b-2)}}{4} = \frac{-b \pm \sqrt{b^2 - 8b + 16}}{4} = \frac{-b \pm |b-4|}{4}$. Since $b > 4$, $m = \frac{-b \pm (b-4)}{4} = \frac{-4}{4}$ or $\frac{-2b+4}{4}$. For $b > 4$, $y = -\frac{b}{2} + 1$ lies below $y = -1$, so the maximum value of m is $\boxed{-1}$.

3. The distance to the circle is just the distance to the center minus the radius. This gives $\left(\sqrt{(x+1)^2 + y^2} - 1\right) + \left(\sqrt{(x-4)^2 + y^2} - 2\right) = 7 \to \sqrt{(x-4)^2 + y^2} = 10 - \sqrt{(x+1)^2 + y^2}$. Squaring and simplifying gives $2x + 17 = 4\sqrt{(x+1)^2 + y^2}$. Squaring and simplifying again gives $273 = 12x^2 - 36x + 16y^2 \to 300 = 12\left(x^2 - 3x + \frac{9}{4}\right) + 16y^2$ which can be written as $1 = \frac{(x - 3/2)^2}{25} + \frac{y^2}{75/4}$. The maximum y-value for this ellipse occurs when $x = 3/2$ and it is $\boxed{y = \frac{5\sqrt{3}}{2}}$.

4. Since $\cos x \neq 0$, then $x \neq \frac{\pi}{2}, \frac{3\pi}{2}$. Setting $\sin x = 0$ we obtain $x = 0, \pi, 2\pi$. Setting $\sin 2x = 0$, we obtain $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, three of which we have and two of which we can't use. Setting $\sin 4x = 0$, we obtain $4x = 0 + \pi k$, making $x = \frac{\pi}{4} + \frac{\pi k}{4}$ and we pick up the following values: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, making for $\boxed{7}$ solutions.

5. Let the sides be x feet and y feet. The reduced dimensions of the rectangle are $x - \frac{1}{12}$ and $y - \frac{1}{12}$. Then $xy - \left(x - \frac{1}{12}\right)\left(y - \frac{1}{12}\right) = 2 \rightarrow xy - \left(xy - \frac{x}{12} - \frac{y}{12} + \frac{1}{144}\right) = 2$. Simplifying we have $\frac{x+y}{12} = 2 + \frac{1}{144} = \frac{289}{144} \rightarrow \frac{x+y}{2} = \frac{289}{24}$. By the AM-GM $\frac{x+y}{2} \geq \sqrt{xy}$, so $\frac{289}{24} \geq \sqrt{xy}$. Squaring gives the largest possible value of the area xy which is $\left(\frac{289}{24}\right)^2$. Thus, the answer is $\boxed{(289, 24)}$.

6. Add 1 to both sides to obtain $xy + x + y + 1 = N + 1 \rightarrow (x+1)(y+1) = N + 1$. If $N + 1$ is prime, then either $x = 0$ or $y = 0$ and so they aren't both positive. Since x and y are both positive, $N + 1$ is not prime. For $x + y$ to have a unique sum, $N + 1$ must have exactly two pairs of factors, one of which is 1 and $N + 1$. The other pair will yield the unique sum. For example, if $N = 34$, then $N + 1 = 35$ which factors into 1 and 35 and 5 and 7. We can't use 1 and 35 since 1 makes x or y equal to 0. If we set $x + 1 = 5$ and $y + 1 = 7$ or $x + 1 = 7$ and $y + 1 = 5$, we obtain the unique sum $x + y = 10$. So we need to count the number of terms in $\{2, 3, 4, \ldots, 50, 51\}$ that have exactly two pairs of factors. These will be the product of two primes. We have 9 numbers in $\{2 \cdot 2, 2 \cdot 3, 2 \cdot 5, \ldots, 2 \cdot 23\}$, 7 numbers in $\{3 \cdot 3, 3 \cdot 5, 3 \cdot 7, \ldots, 3 \cdot 17\}$, 2 numbers in $\{5 \cdot 5, 5 \cdot 7\}$, and 1 number in $\{7 \cdot 7\}$ for a total of $\boxed{19}$ numbers.