LINCOLN-SUDBURY - L

LEAGUES

MAML

STATE INVITATIONAL MATH LEAGUE COMPETITION

April 1, 2011

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2011

Round 1 Arithmetic and Number Theory

1. ____________________

2. ____________________

3. ____________________

1. Determine the positive difference between the mean and median of the first six two-digit prime numbers.

2. The product of all the positive factors of 10,000,000 equals 10^n. Find n.

3. Some 4-digit natural numbers with no zeros can be rearranged to produce exactly 5 other 4-digit numbers. What is the largest possible such 4-digit number for which the sum of its six different arrangements is 23331.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2011

Round 2 **Algebra 1**

1. ______________

2. (,)

3. ______________

1. Sami visits her aunt who lives four miles away. She runs the first mile at a rate of 6 mph. She bicycles the next two miles at a rate of 10 mph and walks the rest of the way. If the entire trip takes Sami 52 minutes, what was her walking rate in miles per hour?

2. Toni has a collection of nickels and dimes with a total value of $2.60. If half the dimes were exchanged for nickels and a third of the original nickels exchanged for dimes, the new value would be $2.65. How many nickels and dimes did Tony have? Express the answer as the ordered pair (nickels, dimes).

3. Solve for x over the set of real numbers: $\dfrac{\sqrt{x-1}+\sqrt{4-x}}{\sqrt{x}+\sqrt{3-x}} < 1$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2011

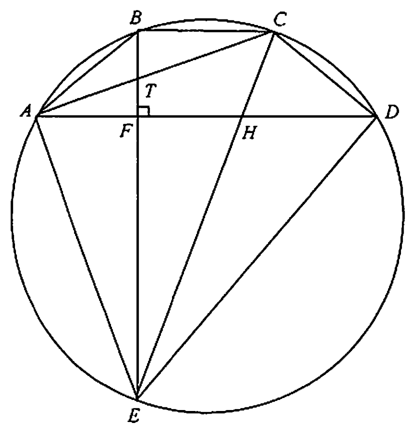
Round 3 – Geometry

1. ______________________

2. ______________________

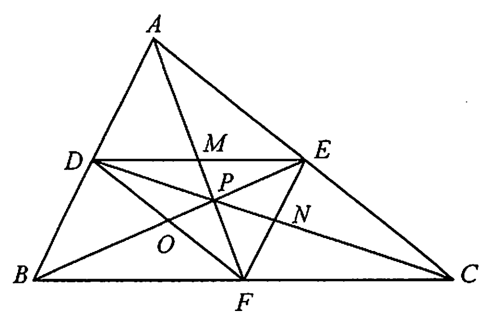
3. ______________________

1. Points B and C trisect minor arc $\overset{\frown}{AD}$ and $\overline{BE} \perp \overline{AD}$. Using just the line segments in the diagram and not counting ΔABT, how many triangles are similar to ΔABT?

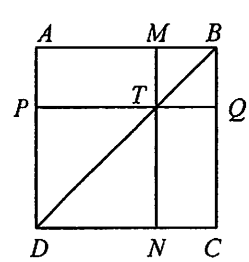


2. In ΔABC, the sum of the lengths of the medians $\overline{AF}$, $\overline{BE}$, and $\overline{CD}$ is 108.

   Find the value of $MP + NP + OP$.



3. In square $ABCD$, $AB = 26$, $\overline{MN}$ and $\overline{PQ}$ are parallel to the sides of the square and intersect at T on $\overline{DB}$. The area of $TQCN$ is 50. Find the length of $\overline{AT}$.



MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2011

Round 4 – Algebra 2

1. ______________________

2. ______________________

3. ______________________

1. If $(4x)^{-3/2} = 81^{3/4}$, determine the value of x.

2. The numbers 10, 11, 12, 13, 21, 22, 23, and 24 are to be placed at the vertices of regular octagon $ABCDEFGH$ such that no two prime numbers are opposite each other. In how many distinct ways can this be done? Note 1: if one arrangement can be turned into another by rotating the octagon, consider those two arrangements to be the same. Note 2: Opposite vertices are the ends of diagonals $\overline{AE}, \overline{BF}, \overline{CG}, \overline{DH}$.

3. For $a \neq 0$, if the real roots of $x^2 + ax + a = 0$ are $\log_3 n$ and $\log_9 n$, find n.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2011

Round 5 – Analytic Geometry

1. (, ,)

2. ________

3. ________

1. The graph of quadratic function $f(x) = ax^2 + bx + c$ has its vertex on the positive y-axis and it intersects the x-axis. The y-intercept is the smallest two-digit prime number and one of its x-intercepts is the largest one-digit prime number. Find the ordered triple (a, b, c).

2. Let P be the origin and let point T lie on a circle with center (4, 0) and radius 1 such that $\overline{PT}$ is tangent to the circle and has positive slope. What is that slope?

3. Ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ intersect at A, B, C, and D. If the area of quadrilateral $ABCD$ equals 484, find the value of $\frac{1}{a^2} + \frac{1}{b^2}$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2011

Round 6 – Trig and Complex Numbers

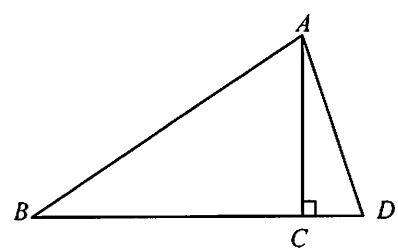
1. ______________________

2. ______________________

3. (, ,)

1. Find the value of $Cos^{-1}\left(\sin\frac{\pi}{5}\right)$

2. $\overline{AC} \perp \overline{BD}$, $AC = 11$, $CD = 1$, and $m\angle B = 2m\angle CAD$. Find the value of BC.



3. Let the complex roots of $z^3 = 1$ be denoted by r_1, r_2, and r_3. Define $p(w) = w^3 + aw^2 + bw + c$ to be the polynomial whose roots are $(r_1 + r_2)^3$, $(r_1 + r_3)^3$, and $(r_2 + r_3)^3$. Determine (a, b, c).

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2011

Team Round

1. ________ : ____________ **4.** ____________________

2. ________________________ **5.** ____________________

3. ________________________ **6.** ____________________

1. Circles centered at P and Q have integer radii and $PQ = 21$. If the ratio of the length of the common external tangent to the length of the common internal tangent is $\sqrt{2}$, determine the ratio of the larger radius to the smaller.

2. In how many ways can the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 be placed on the vertices of a regular decagon, (polygon with 10 sides) one number per vertex, such that there are never three or more consecutive primes. Consider two arrangements the same if one can be rotated into the other.

3. How many integers from 5005 to 6006 inclusive have the property that the sum of the digits in the thousand's and ten's places equals the sum of the digits in the hundred's and one's places?

4. Molly (M) and Nellie (N) start at the indicated squares. Molly goes down 1 square and then, with equal probability, 1 square to the left or right. In the next step she does the same and arrives at a square on the bottom row. Nellie goes up 1 square and then, with equal probability, 1 square to the right or left. In the next step she repeats and ends up on a square in the top row. The *distance* between Molly and Nellie is the minimum number of squares it takes to go from one to the other if one goes only vertically and horizontally.

The distance between them before they travel is 7 as shown. Determine the expected value of the distance between Molly and Nellie when they have reached the bottom and top rows respectively.

		M							
							N		

5. Let ${}_nB_k$ = the number of ways to break a set of *n* elements in *k* disjoint non-empty sets. For example, ${}_3B_2 = 3$ since $\{a,b,c\}$ can be broken into 2 disjoint sets in the following ways: $\{a,b\} \cup \{c\}$, $\{a,c\} \cup \{b\}$, and $\{b,c\} \cup \{a\}$. Note that order doesn't matter. Compute ${}_7B_4$.

6. Today the British pound dropped 30% against the dollar and the Chinese yuan rose 30% against the dollar. The result is that the pound is now worth 1 cent more than the yuan. If, in pennies, yesterday's values were integers, and the pound is now less than 5 cents away from the value of the dollar, find the value of yesterday's pound in cents.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2011

Answer Sheet

Round 1

1. $\frac{2}{3}$
2. 224
3. 6611

Round 2

1. 2
2. $(24,14)$
3. $1 \le x < 2$

Round 3

1. 4
2. 18
3. 24

Round 4

1. $\frac{1}{36}$
2. 2880
3. $\frac{1}{27}$

Round 5

1. $\left(-\frac{11}{49}, 0, 11\right)$
2. $\frac{\sqrt{15}}{15}$
3. $\frac{1}{121}$

Round 6

1. $\frac{3\pi}{10}$
2. 60
3. $(3,3,1)$

Team

1. $\frac{3}{2}$
2. 259,200
3. 76
4. 7
5. 350
6. 137

Solutions: State Meet 2011

Round 1 Arithmetic and Number Theory:

1. The numbers are 11, 13, 17, 19, 23, 29. The mean is $18\frac{2}{3}$. The median is 18.

2. $10{,}000{,}000 = 10^7 = 2^7 \cdot 5^7$. Since 10^7 has $8 \cdot 8 = 64$ factors and since there are 32 factor pairs whose product is 10^7, then the product of all the factors of 10^7 is $\left(10^7\right)^{32} = 10^{224}$. Hence $\boxed{n = 224}$.

3. If the digits were all distinct there would be $4! = 24$ different arrangements. If the number were of the form AABB there would be $\frac{4!}{2! \cdot 2!} = 6$ distinct arrangements. If the number were of the form AAAB there would be 4 arrangements, so we have AABB with A, B $\neq 0$. Given the arrangements AABB, ABAB, ABBA, BABA, BAAB, and BBAA, we note that each digit is in each place three times so the sum of the six numbers equals (3A + 3B) + (3A + 3B)10 + (3A + 3B)100 + (3A + 3B)1000 = 3333(A + B) = 23331. Thus, A + B = 7. The largest possible 4-digit number is $\boxed{6611}$.

Round 2 Algebra I:

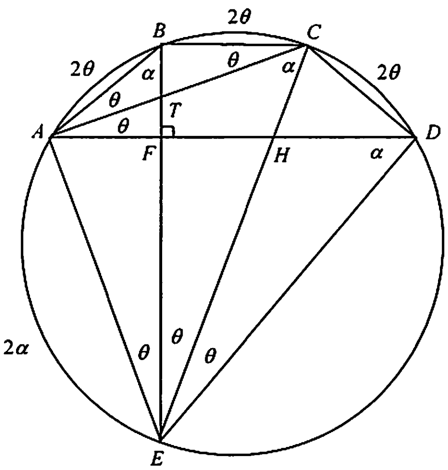
1. $\frac{1\,\text{mi}}{6\,\text{mph}} + \frac{2\,\text{mi}}{10\,\text{mph}} + \frac{1\,\text{mi}}{x\,\text{mph}} = \frac{52\,\text{min}}{60\,\text{min/hr}}$. 10 min + 12 min + x min = 52 min. ∴ x = 30 min. Rate = 2 mph.

2. Solve the system $\begin{cases} 5n + 10d = 260 \\ 5\left(\frac{1}{2}d + \frac{2}{3}n\right) + 10\left(\frac{1}{3}n + \frac{1}{2}d\right) = 265 \end{cases}$. Solving the system gives $(n, d) = (24, 14)$

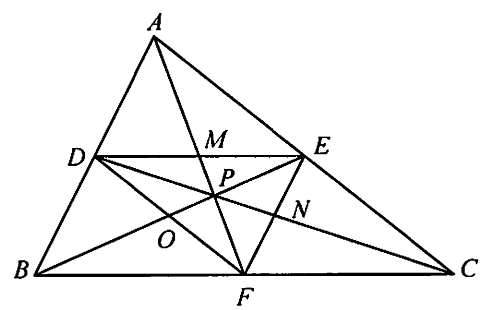
3. The domain is $1 \le x \le 3$. Multiply both sides by $\sqrt{x} + \sqrt{3 - x}$ and square, obtaining $(x - 1) + 2\sqrt{x - 1}\sqrt{4 - x} + (4 - x) < x + 2\sqrt{x}\sqrt{3 - x} + 3 - x$ which simplifies to $\sqrt{x - 1}\sqrt{4 - x} < \sqrt{x}\sqrt{3 - x}$. Square again: $-x^2 + 5x - 4 < -x^2 + 3x \rightarrow 2x < 4$. Combined with the domain restriction we have $\boxed{1 \le x < 2}$.

Round 3 Geometry:

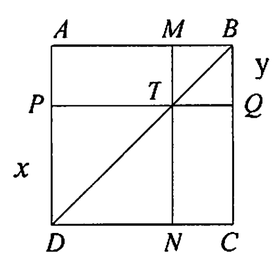
1. Let $m\overset{\frown}{AB} = m\overset{\frown}{BC} = m\overset{\frown}{CD} = 2\theta$ and let $m\overset{\frown}{AE} = 2\alpha$. Then the angles of the triangles are as marked. In ΔABT we have angles of θ and α, so we go looking for other triangles with the same angles. We find them in ΔACH, ΔEBA, ΔECT, and ΔEDH. Answer: $\boxed{4}$.



2. Since M is the midpoint of $\overline{DE}$, then $MF = \frac{1}{2}AF$ and since $\overline{FM}$ is a median of ΔDEF, then $MP = \frac{1}{3}MF \rightarrow MP = \frac{1}{6}AF$. Likewise, $NP = \frac{1}{6}CD$ and $OP = \frac{1}{6}BE$. So $MP + NP + OP = \frac{1}{6} \cdot 108 = \boxed{18}$.



3. The area of $ABCD$ is 676 and since the areas of $AMTP$ and $TQCN$ both equal 50, the areas of $PTND$ and $MBQT$ sum to 576. However, each is a square of side x and y, respectively. Thus, $x^2 + y^2 = 576$ = $PT^2 + AP^2 = 576$. Thus, $AT = \sqrt{PT^2 + AP^2}$ = $\sqrt{576} = \boxed{24}$.



Round 4 Algebra II:

1. $4x = 27^{-2/3} = \frac{1}{9} \rightarrow x = \frac{1}{36}$

2. The first prime can go anywhere, the second prime can go in any one of 6 places and the third prime can go in any one of 4 places. The remaining 5 numbers can be permuted in 5! ways so the answer is $6 \cdot 4 \cdot 5! = 4 \cdot 6! = 4 \cdot 720 = \boxed{2880}$.

3. Using the fact that the coefficient of x equals the negative of the sum of the roots gives $a = -(\log_3 n + \log_9 n) = -\left(\frac{\log n}{\log 3} + \frac{\log n}{\log 9}\right) = -\log n\left(\frac{\log 9 + \log 3}{\log 3 \cdot \log 9}\right)$. Using the fact that the constant term equals the product of the roots gives $a = \log_3 n \cdot \log_9 n = \frac{\log n}{\log 3} \cdot \frac{\log n}{\log 9}$. From $-\log n\left(\frac{\log 9 + \log 3}{\log 3 \cdot \log 9}\right) = \frac{\log n}{\log 3} \cdot \frac{\log n}{\log 9}$ we obtain $-\log 27 = \log n \to \log\frac{1}{27} = \log n$. Thus, $\boxed{n = \frac{1}{27}}$.

Alternate Solution: Let the solutions be r and s. $r = \log_3 n$ and $s = \log_9 n$. So $n = 3^r = 9^s = 3^{2s}$. $\therefore r = 2s$. $x^2 + ax + a = (x - s)(x - 2s) = x^2 - 3sx + 2s^2$. $\to a = -3s = 2s^2 \to s = -\frac{3}{2}, n = 9^s = \frac{1}{27}$

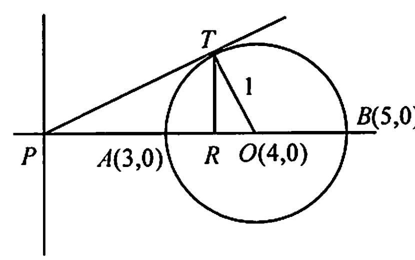
Round 5 Analytic Geometry:

1. The y-intercept is 11 so $c = 11$. The x-intercepts are at 7 and –7. Solve the system $\begin{cases} 0 = 49a + 7b + 11 \\ 0 = 49a - 7b + 11 \end{cases}$

Alternate Solution: Since the vertex is on the y-axis, $b = 0$. $y = a(x+7)(x-7) \to -49a = 11 \to a = -\frac{11}{49}$.

Then $y = -\frac{11}{49}(x^2 - 49)$ giving $\left(-\frac{11}{49}, 0, 11\right)$

2. From $TO^2 = OR \cdot OP$ we obtain $1 = OR \cdot 4 \to OR = \frac{1}{4}$, making $TR = \frac{\sqrt{15}}{4}$ and $PR = \frac{15}{4}$. The slope of $\overline{PT}$ equals $\boxed{\frac{\sqrt{15}}{15}}$. Note that this is the reciprocal of the length of $\overline{PT}$.



Alternate solution 1: Let the equation of PT be $y = mx$. Since it is tangent to the circle whose equation is $(x-4)^2 + y^2 = 1$, $(x-4)^2 + (mx)^2 = 1$ must have a unique solution so we set the discriminate of $(1 + m^2)x^2 - 8x + 15 = 0$ equal to 0 and obtain $64 - 4 \cdot 15(1 + m^2) = 0$ giving $1 = 15m^2 \to m = \frac{1}{\sqrt{15}}$.

Alternate Solution 2: $m = \frac{RT}{PR} = \frac{OT}{PT} = \frac{1}{\sqrt{15}}$ by similar triangles

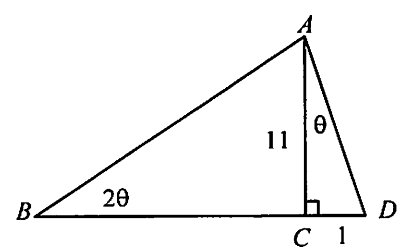
3. Multiply the first equation by $\frac{1}{b^2}$, the second by $\frac{1}{a^2}$ obtaining $\frac{x^2}{a^2b^2}+\frac{y^2}{b^4}=\frac{1}{b^2}$ and $\frac{x^2}{a^2b^2}+\frac{y^2}{a^4}=\frac{1}{a^2}$. Subtracting the second from the first gives $y^2\left(\frac{1}{b^4}-\frac{1}{a^4}\right)=\frac{1}{b^2}-\frac{1}{a^2}\rightarrow$ $y^2\left(\frac{1}{b^2}+\frac{1}{a^2}\right)=1$. Since $ABCD$ is a square of side 22, then $2y = 22$, so $121\left(\frac{1}{b^2}+\frac{1}{a^2}\right)=1$ giving $\frac{1}{a^2}+\frac{1}{b^2} = \boxed{\frac{1}{121}}$.

Alternate solution: Through symmetry, we see that the two ellipses must intersect at four points (A, B, C, and D), one in each quadrant, each of which lies either on the line $y=x$ or $y=-x$. Let A be the point in the first quadrant, and call its coordinates $A(x,x)$. Then since A lies on both ellipses, its coordinates satisfy both equations, and we obtain: $\frac{x^2}{a^2}+\frac{x^2}{b^2}=1 \rightarrow x^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=1 \rightarrow \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{x^2}$. Since $ABCD$ is a square with side length $2x$, the area of $ABCD$ is $4x^2=484 \rightarrow x^2=121$, so $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{121}$.

Round 6 Trigonometry and Complex Numbers:

1. Let $x=\cos^{-1}\left(\sin\frac{\pi}{5}\right)$. Then $\cos x=\sin\left(\frac{\pi}{5}\right)$, so $x=\frac{\pi}{2}-\frac{\pi}{5}=\frac{3\pi}{10}$

2. $\tan\theta=\frac{CD}{AC}=\frac{1}{11}$. $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$ $=$ $\frac{2/11}{1-1/121}=\frac{11}{60}$. Also since $\tan 2\theta=\frac{AC}{BC}=\frac{11}{BC}$, then $\boxed{BC=60}$.

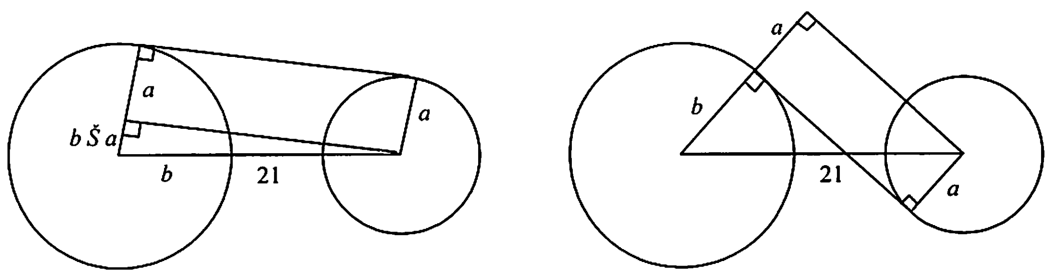


Alternate Solution: $m\angle D=m\angle BAD=90-\theta$ so ΔBAD is isosceles. Let $BC=t$ and $AB=1+t$. Now $(1+t)^2=t^2+121$ and $2t=120$. $\therefore t=BC=60$

3. The roots are 1, $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, and $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$. Adding in pairs gives $\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $\frac{1}{2}-\frac{\sqrt{3}}{2}i$, and $-1+0i$. The first has a 60° angle associated with it, the second a 300° angle, the third a 180° angle, and the moduli are all 1. Cubing gives complex numbers with a modulus of 1 and an angle of 180°, so the cubes all equal -1. Thus, $p(z)=(z+1)^3=z^3+3z^2+3z+1$. Ans: $\boxed{(3,3,1)}$.

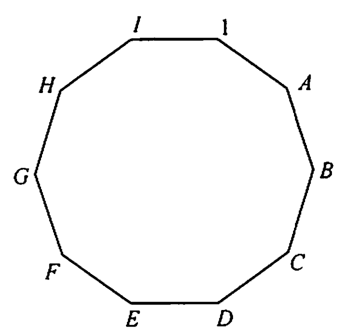
Team :

1. Let the radius of the larger circle be b and the radius of the smaller circle be a. Then as shown in the diagram, the external tangent's length is $\sqrt{21^2 - (b-a)^2}$ and the internal tangent's length is $\sqrt{21^2 - (b+a)^2}$. Thus, we have the following ratio:



$$\frac{\sqrt{21^2 - (b-a)^2}}{\sqrt{21^2 - (b+a)^2}} = \sqrt{2} \rightarrow 21^2 = b^2 + 6ab + a^2.$$ We seek two integers a and b such that $441 = a^2 + 6ab + b^2 = (a+b)^2 + 4ab$. One must be odd and the other even and they are bounded by $1 \le a, b \le 20$. We find that $a = 6, b = 9$ works so $\boxed{\frac{b}{a} = \frac{3}{2}}$.

2. Place 1 somewhere, then the remaining numbers could be placed clockwise in 9! ways. Let's subtract the number of ways that three or more primes could be placed consecutively.



1) Let there be 4 consecutive primes. They can be placed in any one of 6 sequences: $A-D, B-E, \ldots, F-I$. They can be permuted in $4!$ and the remaining 5 composite numbers can be permuted in $5!$ ways, giving $6 \cdot 4! \cdot 5!$ ways for there to be 4 consecutive primes.

2) Let there be 3 consecutive primes. There are two cases:

a) If the three primes are in placed in $A-C$ or $G-I$, then D and F respectively can't be used for a prime, leaving 5 places for a prime. There are two cases, there are 4 choose 3 ways to pick a prime, the primes can be permuted in $3!$ ways, there are 5 places for the last prime and the remaining five composites can be permuted in $5!$ ways. Total: $2 \cdot {}_4C_3 \cdot 3! \cdot 5 \cdot 5!$.

b) If the three primes are placed in $B-D$, $C-E$, $D-F$, $E-G$, or $F-H$ there are only 4 places to put the fourth prime. Thus, we have $5 \cdot {}_4C_3 \cdot 3! \cdot 4 \cdot 5!$ ways to have three primes in a row.

The final answer is $9! - 6 \cdot 4! \cdot 5! - 2 \cdot 4 \cdot 6 \cdot 5 \cdot 5! - 5 \cdot 4 \cdot 6 \cdot 4 \cdot 5! =$
$6!\left(9 \cdot 8 \cdot 7 - 4! - 8 \cdot 5 - 5 \cdot 4 \cdot 4\right) = 8 \cdot 6!(63 - 3 - 5 - 10) = 720 \cdot 360 = \boxed{259{,}200}$.

3. If a number is divisible by 11 the sum of the digits in the odd places of the number minus the sum of the digits in the even places is a multiple of 11. The number of multiples of 11 in the interval [5005, 6006] is $\frac{6006 - 5005}{11} + 1 = 92$. Many of those have the property that the difference between the sum of the digits in the first and third places and the second and fourth places is 0, but some don't. Those have a difference of 11 or –11 and we must eliminate those. If the difference is 11 we have the following:

11–0	5060
12–1	5170, 5071
13–2	5280, 5082, 5181
14–3	5390 5093, 5291, 5192

If the difference is –11 we have the following:

5–16	5907, 5709, 5808
6–17	5918, 5819
7–18	5929

Thus, we must reject 10 + 6 numbers giving $92 - 16 = \boxed{76}$ numbers with the desired property.

4. Molly ends up at either *A* or *C* with a probability of ¼ and *C* with a probability of ½ while Nellie ends up at either *D* or *F* with a probability of ¼ and *E* with a probability of ½. To calculate the expected distance, sum up products consisting of the probability of reaching each endpoint times the distance between those endpoints. For example, if Molly reaches *A* and Nellie reaches *E*, we have (1/4)(1/2)(9) where 9 is the distance from *A* to *E* consisting of 7 horizontal moves and 2 vertical moves.

					D		*E*		*F*
A		*B*		*C*					

Calculating the products for distances *AD, AE, AF, BD, BE, BF, CD, CE,* and *CF* respectively gives:

$$\frac{1}{4} \cdot \frac{1}{4} \cdot 7 + \frac{1}{4} \cdot \frac{2}{4} \cdot 9 + \frac{1}{4} \cdot \frac{1}{4} \cdot 11 + \frac{2}{4} \cdot \frac{1}{4} \cdot 5 + \frac{2}{4} \cdot \frac{2}{4} \cdot 7 + \frac{2}{4} \cdot \frac{1}{4} \cdot 9 + \frac{1}{4} \cdot \frac{1}{4} \cdot 3 + \frac{1}{4} \cdot \frac{2}{4} \cdot 5 + \frac{1}{4} \cdot \frac{1}{4} \cdot 7$$

This sums to $\boxed{7}$.

5. Possibilities are 4-1-1-1, 3-2-1-1, or 2-2-2-1. If 4-1-1-1 we have ${}_7C_4 = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ ways to pick elements for the set containing 4 elements. The other three sets consisting of one element are determined and so there is no need to count them. If 3-2-1-1 we have

${}_7C_3 \cdot {}_4C_2 = 35 \cdot \frac{4 \cdot 3}{2 \cdot 1} = 210$ ways to pick the elements for a set of three elements and two elements for the set of two elements. The two sets of one element are determined so there is no need to count them. If we have 2-2-2-1, then we have ${}_7C_2 \cdot {}_5C_2 \cdot {}_3C_2 = \frac{7 \cdot 6}{2 \cdot 1} \cdot \frac{5 \cdot 4}{2 \cdot 1} \cdot 3 = 630$. However, this counts the ordering of the three sets of 2 so we must divide 630 by 3!, obtaining 105. Total: $35 + 210 + 105 = \boxed{350}$.

6. Let x = the value of the pound yesterday in pennies and let y = the value of the yuan yesterday in pennies. Then $.7x = 1.3y + 1 \rightarrow 7x = 13y + 10$. We seek solutions for x such that $.7x$ lies between 95 pennies and 105 pennies. If $y = 3$ and $x = 7$ the equation is satisfied but $.7x$ doesn't lie between 95 and 105. Experience with these equations teaches that the x-values increase by 13 while the y-values increase by 7. Thus, solutions are of the form $y = 7t + 3$ and $x = 13t + 7$. Solving $95 < \frac{7}{10}(13t + 7) < 105$ gives $901 < 91t < 1001$. Thus, $t = 10$, making $x = \boxed{137}$.

Alternate solution: From $7x = 13y + 10$ we obtain $7x - 10 = 13y \rightarrow (7x + 3) - 13 = 13y$. So 13 must divide $7x + 3$, making $x = 7, 20, 33, \ldots$. From $95 < .7x < 105$ we obtain $950 < 7x < 1050$, making $x = 136, 137, 138, \ldots, 149$. Only 137 lies in the sequence 7, 20, 33, . . . so the answer is 137.