MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2012

Round 1 Arithmetic and Number Theory

1. ______________________

2. ______________________

3. ______________________

1. A = the median of the first 12 prime numbers,
   B = the mean of the first 5 prime numbers larger than 50.
   What is the third prime number larger than $A + B$?

2. Each of 1354 families has one car, one TV, and one phone. There are 4 kinds of cars, 3 kinds of TV's, and 8 kinds of phones. You can guarantee that at least n families own the same type of car, TV and phone. What is the value of n?

3. Start with a three-digit number. Double the digit in the hundred's place, carrying if necessary. Then double the digit in the ten's place, carrying if necessary. Then double the digit in the one's place, carrying if necessary. Then double the digit in the hundred's place, carrying if necessary. Let the result be N. Determine the number of 3-digit numbers for which it is true that $100 \le N \le 1000$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2012

Round 2 Algebra 1

1. ______________________

2. ______________________

3. ______________________

1. The news today stated that 1 in 11 people in their fifties do not get enough food. That's about 80% higher than 10 years ago. If the rate was 1 in n ten years ago, find the closest integer value for n.

2. Define $\lfloor x \rfloor$ to be the greatest integer less than or equal to x. Determine the values of x for which $\lfloor |x| \rfloor = |\lfloor x \rfloor|$ on the closed interval $[-4, 4]$.

3. Find all values of k such that all of the following equations have real solutions:

$$kx^2 + x + 1 = 0,\ x^2 + kx + 1 = 0,\ x^2 + x + k = 0$$

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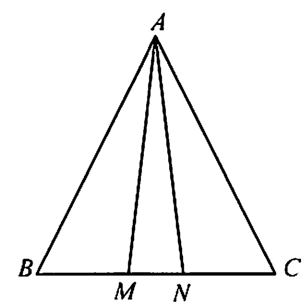
Round 3 – Geometry

1. ______________________

2. ______________________

3. ______________________

1. In ΔABC, $\overline{AM}$ and $\overline{AN}$ trisect $\angle BAC$, and $AB = AC$. If $BM = 2(MN)$ and $AM = 5$, find the number of units in the length of $\overline{AB}$.



2. By cutting congruent 45-45-90 right triangles out of the corners of a square of side 8, an octagon is created. Find the number of units in the perimeter of the octagon if its area is three-fourths that of the square.

3. The sides of a regular 24-gon are extended to produce a star with 24 vertices. Find the maximum number of degrees in the sum of the measures of the 24 vertex angles.

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STATE PLAYOFFS – 2012

Round 4 – Algebra 2

1. ____________________

2. ____________________

3. ____________________

1. Determine the number of functions that are not 1–to–1 and that map from a 4 element set to a 5 element set.

2. Let a, b, and r be real numbers with infinite series of nonzero terms $a + ar + ar^2 + \cdots = 2$ and $b + br + br^2 + \cdots = 5$, where $a + b = K$. Find all possible values of K.

3. The side of a square is $\log_{10} a^x$ and the perimeter of the square is $\log_{10} b^y$. If x and y are positive integers with $x + y = 12$, determine the largest possible value of $\log_a b$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2012

Round 5 – Analytic Geometry

1. ______________________

2. ______________________

3. ______________________

1. A circle of radius 1 is tangent to both arms of the graph of $y = -|x|$ and a circle of radius 3 is tangent to both arms of the graph of $y = |x - 2|$. Determine the number of units in the distance between the centers.

2. Let $ABCD$ be the quadrilateral whose vertices are $A(0, 0)$, $B(2, 6)$, $C(10, 10)$, and $D(8, 4)$. Find the slope of the line passing through $P(3, 7)$ that divides the quadrilateral into two parts, each having the same perimeter.

3. Three of the vertices of a regular hexagon have coordinates $(-3,1)$, $(-1, 1 + 2\sqrt{3})$, and $(3, 1 + 2\sqrt{3})$. Find the number of square units in the area of this hexagon.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2012

Round 6 – Trig and Complex Numbers

1. ____________________

2. ____________________

3. ____________________

1. For x in radians, solve $\left(\text{Tan}^{-1}\left(\frac{1}{x}\right)\right)^{-1} = 2$. Express the value of x in terms of the cosecant function.

2. Let t be in radians. Find the number of solutions to $(\cos t)(\sin t) = \frac{1}{3}\cos t$ in $[0, 2011\pi]$.

3. The position of a particle is given by $x = \frac{1}{2}\sin^2 t$ and $y = 2\cos^2 t$.
How far does the particle move for $0 \le t \le 2012\pi$?

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2012

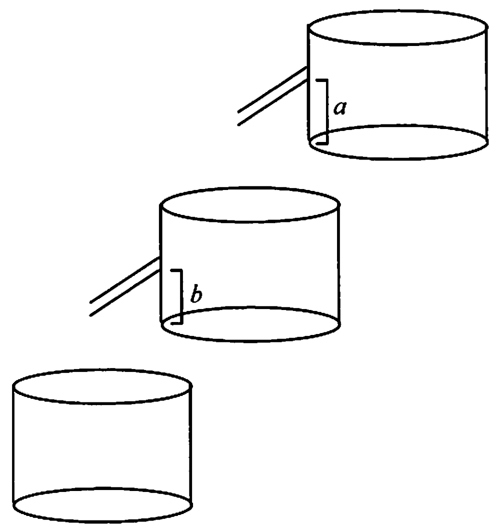
Team Round

1. ______________________ **4.** ______________________

2. ______________________ **5.** ______________________

3. ______________________ **6.** ______________________

1. The diagram shows 3 congruent cylinders, each with a height of 20 cm and a radius of 6 cm. The spout of the top cylinder is sealed on the inside and the cylinder is then filled with water. The bottoms of the spouts in the top two cylinders are a cm and b cm above the bases, respectively, where a and b are integers such that $1 \le a \le 19$, $1 \le b \le 19$ and $a > b$. The seal in the top cylinder is removed. When the water stops flowing the height of the water in the bottom cylinder is n cm where n is a positive integer. Determine the number of possible ordered pairs (a,b).



2. For $0 \le \theta \le 2\pi$, in how many distinct points do the graphs of

$r_1 = 1 + 2\cos\theta$ and $r_2 = \dfrac{1}{1 + 2\cos\theta}$ intersect?

3. Xenia would like to send the following text message: *if ace be hid egad.* Given the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, she can assign them to three buttons, three distinct letters to a button, in any way that she wishes. A letter in the first position requires one push of the button, a letter in the second requires 2 pushes, and a letter in the third requires 3. If she assigns letters so that the least number P of pushes occurs, find the value of P.

4. Find the value of the product abc given the three numbers a, b, and c such that

$$\begin{aligned} a + b + c &= 8 \\ a^2 + b^2 + c^2 &= 18 \\ a^4 + b^4 + c^4 &= 100 \end{aligned}$$

5. Given square $ABCD$ with $A = (1,0)$, $B = (0,-1)$, $C = (-1,0)$, and $D = (0,1)$, let point P lie on a side of $ABCD$ and let O be the origin. If point P' lies on an extension of $\overline{OP}$ such that $OP \cdot OP' = 1$, find the greatest distance of P' from the origin.

6. Determine the number of pairs of points P and Q that lie on the graph of $y = x^2$ such that the chord $\overline{PQ}$ passes through $A(0, 144)$ and the coordinates of P and Q are integers.

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Answer Sheet

Round 1

1. 89
2. 15
3. 151

Round 2

1. 20
2. $-4, -3, -2, -1,\ 0 \le x \le 4$

   or $-4, -3, -2, -1, [0, 4]$
3. $k \le -2$

Round 3

1. $AB = 10$
2. $48 - 16\sqrt{2}$ or $16\left(3 - \sqrt{2}\right)$
3. 3600

Round 4

1. 505
2. $0 < k < 14,\ k \ne 7$
3. 44

Round 5

1. 6
2. -1
3. $24\sqrt{3}$

Round 6

1. $\sqrt{\csc^2\left(\frac{1}{2}\right) - 1}$
2. 4023
3. $2012\sqrt{17}$

Team

1. 81
2. 7
3. 24
4. $\frac{417}{16}$
5. $\sqrt{2}$
6. 15

Solutions: State Meet 2012

Round 1 Arithmetic and Number Theory:

1. The median of the first twelve primes is the mean of the 6th and 7th prime, 15. The mean of 53, 59, 61, 67, and 71 is 62.2. 62.2 + 15 = 77.2. The third prime after that is $\boxed{89}$.

2. There are 4(3)(8) = 96 possible combinations of cars, TV's, and phones. Using the pigeonhole principle, if families are distributed equally among the possible combinations, then the least possible number with the same combination is $\frac{1354}{96} = 14\frac{10}{96} \Rightarrow \boxed{15}$

3. We note that no numbers from 300 on up can give a result between 100 and 1000 since doubling 3 twice would give a result at least as large as 1200. Numbers between 251 and 300 would result in a number greater than 500 after the digit in the hundred's place and the digit in the ten's place were doubled. The final doubling would push the result over 1000. With 250 we have $250 \to 450 \to 500 \to 1000$, so 250 is the largest number that gives a result in the desired interval. $250 - 100 + 1 = \boxed{151}$.

   Additional explanation: We note that $249 \to 449 \to 489 \to 498 \to 898$. We realize that the result will always be even. We also note that $200 \to 400 \to 800$, so we have all the even numbers from 800 to 898. But $199 \to 299 \to 389 \to 398 \to 698$, $150 \to 250 \to 300 \to 600$ while $149 \to 249 \to 289 \to 298 \to 498$. Finally, $100 \to 200 \to 400$. We realize that we don't obtain all even numbers from 400 to 1000, just the even numbers from 400 to 498, 600 to 698, and 800 to 898 and 1000.

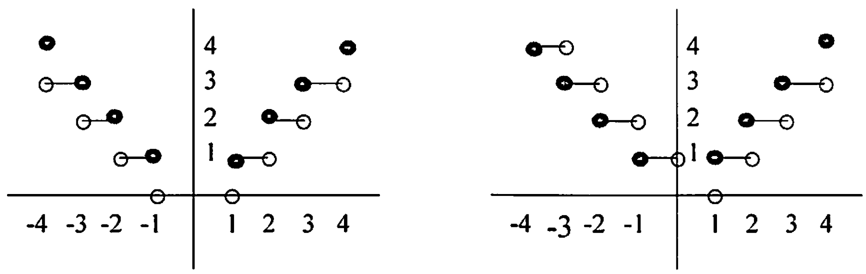
   From 00 to 98 there are $\frac{98-0}{2} + 1 = 50$ numbers, the answer is $3 \cdot 50 + 1 = \boxed{151}$.

Round 2 Algebra I:

1. $x + .8x = \frac{1}{11} \rightarrow x = \frac{5}{99} = .\overline{05}$. This is very close to 1 out of 20.
   Thus, the best value for n is $\boxed{20}$.

   Alternate solution: If we start with 990 people, then 1 in 11 makes 90 people. That is 80% larger than 50 and $\frac{50}{990} = \frac{5}{99} \approx \frac{1}{20}$.

2. Consider the graphs. The functions are the same for $x \geq 0$, but they are offset except at integer values when x is negative. Answer: $\boxed{-4, -3, -2, -1,\ 0 \leq x \leq 4}$.

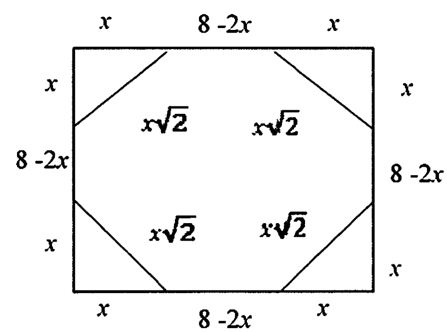


3. Solving, we have $x = \frac{-1 \pm \sqrt{1-4k}}{2k}$, $\frac{-k \pm \sqrt{k^2-4}}{2}$, and $\frac{-1 \pm \sqrt{1-4k}}{2}$ respectively. From the middle expression, we obtain $k^2 - 4 \geq 0 \rightarrow k \leq -2$ or $k \geq 2$. From the other two, we have $1 - 4k \geq 0 \rightarrow k \leq \frac{1}{4}$. Thus, for $\boxed{k \leq -2}$, we obtain real solutions for all three equations.

Round 3 Geometry:

1. Using the Triangle Angle Bisector Theorem on ΔBAN, since $\overline{AM}$ bisects $\angle BAN$, we have $\frac{AB}{BM}=\frac{AN}{MN}\rightarrow\frac{AB}{2x}=\frac{5}{x}$. Thus, $\boxed{AB = 10}$.

Let the legs of the 45-45-90 right triangles be x, giving the equation $8^2 - 4\cdot\frac{1}{2}x^2 = \frac{3}{4}\cdot 64$, $x^2 = 8$, or $x = 2\sqrt{2}$. The vertical and horizontal sides of the octagon equal $8 - 2x$ and the diagonal sides equal $x\sqrt{2}$ so the perimeter is $4(8 - 2x) + 4x\sqrt{2}$ = $4\left(8-2\cdot 2\sqrt{2}\right)+4\left(2\sqrt{2}\cdot\sqrt{2}\right)$ = $\boxed{48-16\sqrt{2}}$ or $\boxed{16\left(3-\sqrt{2}\right)}$.



3. There are 24 triangles formed using the side of the 24-gon as a base and the extensions as legs. The sum of the measures of their angles is $24\cdot 180$. At each vertex of the 24-gon there are two exterior angles which constitute pairs of vertical angles. Each exterior angle measures $\left(\frac{360}{n}\right)^{\circ}$ and, therefore, the two sets of exterior angles total $2\cdot 360°$ and these angles form two of the three angles of each triangle. Thus, the sum of the measures of the vertex angles is
$24\cdot 180 - 2\cdot 360 = 24\cdot 180 - 4\cdot 180 = \boxed{3600}$.

Round 4 Algebra II:

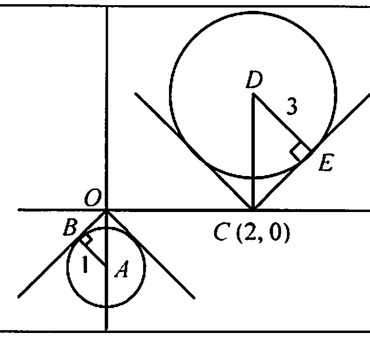
1. Given a domain of $\{a,b,c,d\}$ and a range of $\{M,N,P,Q,R\}$ we have 5 choices for $f(a)$, but we also have 5 choices for $f(b)$ and so on giving $5^4 = 625$ for the number of functions. Since there are $5\cdot4\cdot3\cdot2=120$ one-to-one functions, the answer is $\boxed{505}$.

2. Since both series converge, we have $-1<r<1$, $r\neq0$.
From the first series, $\frac{a}{1-r}=2$ and from the second, $\frac{b}{1-r}=5$.
Thus, $\frac{a}{b}=\frac{2}{5}\rightarrow\frac{2b}{5}+b=K\rightarrow b=\frac{5K}{7}$. Hence $\frac{5K/7}{1-r}=5\rightarrow\frac{K}{7}=1-r$.
Solving for r, $r=1-\frac{K}{7}=\frac{7-K}{7}$. If $r\neq0$, then $K\neq7$ and if $-1<r<1$, then $-1<\frac{7-K}{7}<1\rightarrow-14<-K<0$ or $0<K<14$. Thus, we have $\boxed{0<K<14,\ K\neq7}$.
Alternately, since $a = 2(1-r)$ and $b = 5(1-r)$, we have $a + b = K = 7(1-r)$.
Applying the convergence requirements, $r\neq0$ gives us $K\neq7$ and $-1<r<1\rightarrow2>1-r>0$
Multiplying by 7, we have $14 > K > 0$ and the same results as above.

3. Let t be the side of the square, then $t = x\log a$ and $4t = y\log b$, giving $4x\log_{10}a = y\log_{10}b$, so $\frac{4x}{y} = \frac{\log b}{\log a} \rightarrow \frac{4(12-y)}{y} = \frac{48}{y} - 4 = \log_a b$.

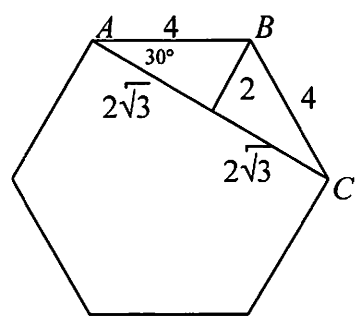
The maximum value occurs when $y = 1$, namely, $\boxed{44}$.

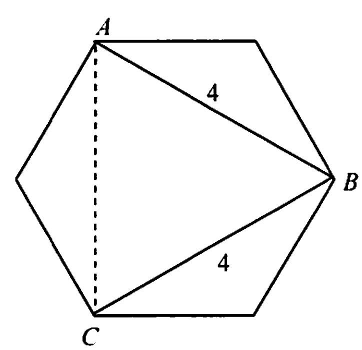
Round 5 Analytic Geometry:

1. Since ΔAOB is a 45-45-90 right triangle where $AB = 1$, we have $AO = \sqrt{2}$, giving $A = \left(0, -\sqrt{2}\right)$. Likewise, $DC = 3\sqrt{2}$, making $D = \left(2, 3\sqrt{2}\right)$. Thus, $AD = \sqrt{(2-0)^2 + \left(3\sqrt{2} - -\sqrt{2}\right)^2} = \sqrt{4+32} = \boxed{6}$.	

2. Looking at the slopes of the sides we note that $m\overline{AB} = 3$, $m\overline{BC} = \frac{1}{2}$, $m\overline{CD} = 3$, and $m\overline{DA} = \frac{1}{2}$, so *ABCD* is a parallelogram. The line from (3, 7) through the intersection of the diagonals of the parallelogram ought to bisect the perimeter since it would divide the parallelogram into two congruent figures. The equation of $\overline{AC}$ is $y = x$ and the equation of $\overline{BD}$ is $y - 6 = -\frac{1}{3}(x-2)$. The two intersect at (5, 5). The slope of the line connecting (5, 5) with (3, 7) is $\frac{7-5}{3-5} = \boxed{-1}$.

3. Given $A(-3,1)$, $B(-1, 1+2\sqrt{3})$, and $C(3, 1+2\sqrt{3})$, we have $AB = \sqrt{2^2 + (2\sqrt{3})^2} = 4$, $BC = 4$, and $AC = \sqrt{6^2 + (2\sqrt{3})^2} = 4\sqrt{3}$. In any regular hexagon *PQRSTU*, there are 6 sides (*PQ*, *QR*, *RS*, *ST*, *TU* and *UP*), and 9 diagonals, 3 long (*PS*, *QT* and *RU*) and 6 short (*PR*, *QS*, *RT*, *SU*, *TP* and *UQ*). Since the long diagonals share no common vertex, *A*, *B* and *C* can not be endpoints of long diagonals. Given $AB = BC = 4$, we consider the two possibilities:
(1) *AB* and *BC* are consecutive sides or (2) *AB* and *BC* are short diagonals sharing a common vertex
The diagram on the right shows that case (2) is impossible, since *AC* would also be a short diagonal and $AC \neq 4$. We consider the left diagram:





Here the side of the hexagon is 4. If we break the hexagon up into 6 equilateral triangles of side 4, we obtain an area equal to $6 \cdot \frac{4^2\sqrt{3}}{4} = \boxed{24\sqrt{3}}$.

Round 6 Trigonometry and Complex Numbers:

1. Taking the reciprocal, $\text{Tan}^{-1}\left(\frac{1}{x}\right) = \frac{1}{2}$. The principal inverse tangent is defined in quadrant 1 (for positive values) and quadrant 4 (for negative values). Thus, $\frac{1}{x}$ denotes a positive value and, consequently $x > 0$. Taking tan of both sides, $\frac{1}{x} = \text{Tan}\frac{1}{2} \rightarrow x = \frac{1}{\text{Tan}\frac{1}{2}} = \cot\frac{1}{2}$.

   Since $1+\cot^2 t = \csc^2 t$, we have $1+x^2 = \csc^2\frac{1}{2}$, giving $x = \pm\sqrt{\csc^2\left(\frac{1}{2}\right)-1}$. However, since $x > 0$, we have $x = \boxed{\sqrt{\csc^2\left(\frac{1}{2}\right)-1}}$.

2. We have $\cos t = 0$ or $\sin t = \frac{1}{3}$. In $[0,2\pi]$, $\cos t = 0$ for $t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. There are two solutions for $\sin t = \frac{1}{3}$ as well, one in $\left[0,\frac{\pi}{2}\right]$ and one in $\left[\frac{\pi}{2},\pi\right]$. Thus, for every period from $[0,2\pi]$ to $[2008\pi, 2010\pi]$ there are 4 solutions making a total of $4 \cdot 1005 = 4020$. In the last interval $[2010\pi, 2011\pi]$, there are 2 solutions to the sine equation, but only 1 for the cosine equation, making a grand total of $\boxed{4023}$.

3. From $2x = \sin^2 t$ and $\frac{y}{2} = \cos^2 t$ we obtain $2x + \frac{y}{2} = 1$. At $t = 0$ the particle is at (0, 2). At $t = \frac{\pi}{2}$ it is at $\left(\frac{1}{2},0\right)$. At $t = \pi$ it is back at (0, 2) and so it travels $\sqrt{(2-0)^2 + \left(\frac{1}{2}-0\right)^2} = \frac{1}{2}\sqrt{17}$ for each time interval of duration $\frac{\pi}{2}$. There are $\frac{2012\pi}{\frac{\pi}{2}} = 4024$ such intervals.

   Thus, the particle moves $\boxed{2012\sqrt{17}}$ units over the specified time period.

Team :

1. In the top container water at a height above a units flows into the middle container. Thus, $(20 - a)$ flows into the middle container. Once the water level in the middle container reaches a height of b units, any additional water flows into the bottom container. Thus, $n = (20 - a) - b$. We make a list: If $a = 19$, then $20 - a = 1$. However, since $a > b$ that would make $b = 0$ and this is not allowed. If $a = 18$, then $20 - a = 2$, $b = 1$ and $n = 1$. If $a = 17$, then b can be 1 or 2 and n can be 2 or 1. We notice that this pattern holds. For example, if $a = 14$, then $20 - a = 6$, and b can be 1, 2, 3, 4, or 5 while n can be 5, 4, 3, 2, 1. However, since $a > b$ we have the following: when $a = 10$, then $20 - a = 10$ and b can take on values from 1 to 9 while n takes on values from 9 to 1. But when $a = 9$ and $20 - a = 11$, b can only take on values from 1 to 8. Likewise, when $a = 8$ and $20 - a = 12$ b can only take on values from 1 to 7. Thus, we have the following table:

a	18	17	16	15	14	13	12	11	10
$20 - a$	2	3	4	5	6	7	8	9	10
b	1	1, 2	1, 2, 3	1 – 4	1 – 5	1 – 6	1 – 7	1 – 8	1 – 9
n	1	2, 1	3, 2, 1	4 – 1	5 – 1	6 – 1	7 – 1	8 – 1	9 – 1

a	9	8	7	6	5	4	3	2	1
$20 - a$	11	12	13	14	15	16	17	18	19
b	1 – 8	1 – 7	1 – 6	1 – 5	1 – 4	1, 2, 3	1, 2	1	–
n	10 – 3	11 – 5	12 – 7	13 – 9	14 – 11	15 – 13	16, 15	17	–

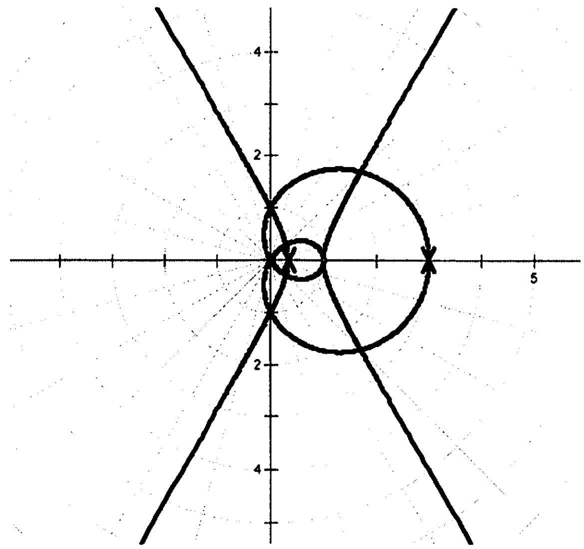
This gives a total of $(1 + 2 + \ldots + 9) + (8 + 7 + \ldots + 2 + 1) = 45 + 36 = \boxed{81}$ ordered pairs (a, b).

2. Setting $2\cos\theta + 1 = \dfrac{1}{2\cos\theta + 1}$ gives $4\cos^2\theta + 4\cos\theta + 1 = 1 \to \cos\theta(\cos\theta + 1) = 0$. Thus, $\cos\theta = 0 \to \theta = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$ or $\cos\theta = -1 \to \theta = \pi$. It would appear that there are 3 solutions. But we've obtained only the simultaneous solutions where the r values and θ values are equal. **The curves may also intersect each other at angles 180° apart such that one *r* value is the negative of the other.** Thus, we set $-(2\cos(\pi + \theta) + 1) = \dfrac{1}{2\cos\theta + 1}$. The left side becomes $-(-2\cos\theta + 1) = 2\cos\theta - 1$ giving $2\cos\theta - 1 = \dfrac{1}{2\cos\theta + 1} \to 4\cos^2\theta = 2 \to$ $\cos\theta = \pm\dfrac{1}{\sqrt{2}} \to \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$, and $\dfrac{7\pi}{4}$. This makes for a total of $\boxed{7}$ points of intersection. In terms of (r, θ) those points are

$\left(1, \dfrac{\pi}{2}\right), (-1, \pi), \left(1, \dfrac{3\pi}{2}\right), \left(\sqrt{2} + 1, \dfrac{\pi}{4}\right), \left(-\sqrt{2} + 1, \dfrac{3\pi}{4}\right), \left(-\sqrt{2} + 1, \dfrac{5\pi}{4}\right)$, and

$\left(\sqrt{2} + 1, \dfrac{7\pi}{4}\right)$. Note that the graph of $r = 1 + 2\cos\theta$ is a limaçon with an interior loop, while the graph of its reciprocal is a hyperbola.

Here are the graphs:



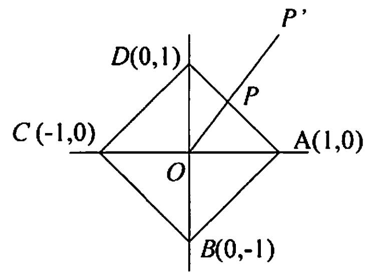
3. Here is the frequency count of the letters:

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
2	1	1	2	3	1	1	1	2

Put three of the most frequently occurring letter (e.g. *A*, *D*, and *E*) in the first position, giving a total of $3 + 2 + 2 = 7$ pushes. Put three of the least frequently occurring letters (e.g. *F*, *G*, and *H*) in the third position, requiring 9 pushes. Finally, the remaining letters *I*, *B*, and *C* can go in the second position, requiring $4 + 2 + 2 = 8$ pushes. Total: $\boxed{24}$

4. We'll use the relationship $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + xz + yz)$ throughout this solution. $(a+b+c)^2 = 8^2 \rightarrow a^2 + b^2 + c^2 + 2(ab + ac + bc) = 64$, Then $ab + bc + ac = 23$ by replacing $a^2 + b^2 + c^2$ with 18. Square $a^2 + b^2 + c^2 = 18$ to get $a^4 + b^4 + c^4 + 2\left((ab)^2 + (ac)^2 + (bc)^2\right) = 324$. Replacing $a^4 + b^4 + c^4$ with 100 gives $(ab)^2 + (ac)^2 + (bc)^2 = 112$. Squaring $ab + bc + ac = 23$ gives $(ab)^2 + (ac)^2 + (bc)^2 + 2\left(a^2bc + ab^2c + abc^2\right) = 529$. Replace $(ab)^2 + (ac)^2 + (bc)^2$ with 112 and factor: $2(abc)(a + b + c) = 417$. Replace $a + b + c$ by $8 \rightarrow 16abc = 417$, so $abc = \boxed{\dfrac{417}{16}}$.

5. Because of symmetry we need only consider the lines as drawn in the first quadrant. Let the coordinates of point P be (x, y). From $OP \cdot OP' = 1$ we obtain $\sqrt{x^2+y^2} \cdot OP' = 1$. OP' will take on its maximum value when $\sqrt{x^2+y^2}$ takes on its minimum value. Letting $y = 1 - x$, consider $x^2 + y^2 = x^2 + (1-x)^2 =$ $2x^2 - 2x + 1$. The parabola takes on its minimum value at $x = -\frac{-2}{2 \cdot 2} = \frac{1}{2}$ and its minimum value is $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$. Thus, the maximum value of OP' is $\frac{1}{\sqrt{\frac{1}{2}}} = \boxed{\sqrt{2}}$.



Alternate Solution: Consider the circle inscribed in the square. The minimum value of OP, where P is a point on the line $\overline{AD}$, must occur at the point of tangency.
Thus, P is the midpoint of $\overline{AD}$, $P = \left(\frac{1}{2}, \frac{1}{2}\right)$; and $OP^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, $OP = \frac{1}{\sqrt{2}}$ so $OP' = \sqrt{2}$.

6. Given $P\left(x_1, x_1^2\right)$ and $Q\left(x_2, x_2^2\right)$, the slope of $\overline{PQ}$ equals $\frac{x_2^2 - x_1^2}{x_2 - x_1} = x_2 + x_1$.
Since A lies on $\overline{PQ}$, the slope of $\overline{PA}$ equals the slope of $\overline{PQ}$ giving
$\frac{144 - x_1^2}{0 - x_1} = x_1 + x_2 \rightarrow 144 - x_1^2 = -x_1^2 - x_1x_2$.
Thus, $x_1x_2 = -144$ and we need to search for factors of 144.
We have the following 8 pairs if $x_1 < 0$:
$P(-1,1)$ & $Q\left(144,144^2\right)$, $P(-2,4)$ & $Q\left(72,72^2\right)$ up to $P(-9,81)$ & $Q\left(16,16^2\right)$.
The last is $P(-12,144)$ & $Q(12,144)$.
We have the following 8 pairs if $x_1 > 0$:
$P(1,1)$ & $Q\left(-144,(-144)^2\right)$, $P(2,4)$ & $Q\left(-72,(-72)^2\right)$ up to $P(9,81)$ & $Q\left(-16,(-16)^2\right)$.
However, the last one in this sequence $P(12,144)$ & $Q\left(-12,(-12)^2\right)$ was previously counted. Thus, the total is $2(8) - 1 = \boxed{15}$.