Judge

LEAGUES

MAML

STATE INVITATIONAL MATH LEAGUE COMPETITION

March 27, 2015

Shrewsbury High School

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2015

Round 1 **Arithmetic and Number Theory**

1. ____________________

2. ____________________

3. ____________________

1. <u>How many</u> prime numbers less than 200 have the property that they are two less than a perfect square?

2. On a number line, point A corresponds to $-\frac{3}{5}$ and point B corresponds to $\frac{7}{8}$.
Point $C$is between A and B andis such that $AC = \frac{2}{3}AB$.
To what number does point C correspond?

3. Find the <u>sum</u> of all positive 3-digit multiples of 3 in which all the digits are odd and exactly two digits are identical.

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Solutions: State Meet 2015

Round 1 Arithmetic and Number Theory:

1. 6: The numbers are 2, 7, 23, 47, 79, 167

2. $AB = \frac{3}{5} + \frac{7}{8} = \frac{59}{40}$; $\frac{2}{3} \cdot \frac{59}{40} = \frac{59}{60}$; $-\frac{3}{5} + \frac{59}{60} = \frac{23}{60}$

3. The possible digit sums for a 3-digit multiple of 3 with odd digits are 3, 9, 15, 21 and 27.
If the identical digits are 1s, the only choice for the third digit is 7. $\{1,1,7\} \Rightarrow 117,\ 171,\ 711 \Rightarrow 999$
If the identical digits are 3s, the only choice for the third digit is 9
$\{3,3,9\} \Rightarrow 339,\ 393,\ 933 \Rightarrow (15)(15)(15) \Rightarrow 1665$
If the identical digits are 7s, the only choice for the third digit is 1.
$\{7,7,1\} \Rightarrow 771,\ 717,\ 177 \Rightarrow (15)(15)(15) \Rightarrow 1665$
If the identical digits are 9s, the only choice for the third digit is 3.
$\{9,9,3\} \Rightarrow 993,\ 939,\ 399 \Rightarrow (21)(21)(21) \Rightarrow 2331$
Identical digits of 5 is not possible, since $\{5,5,x\} \Rightarrow x = 5$, violating the given conditions.
Thus, the required sum is **<u>6660</u>**.

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Round 2 **Algebra 1**

1. ______________________

2. ______________________

3. (________ , ________)

1. If $(3x+2)(6x+1)=0$, then the expression $12x-5$ has what possible value(s)?

2. The numerator and denominator of a fraction are both integers and the denominator is four more than the numerator. If the reciprocal of the fraction is added to the fraction and the result is divided by the difference between the fraction and 1, the resultis $\frac{13}{2}$. There are <u>two</u>possible expressions forthe fractionin $\frac{x}{x+4}$ form. Determine the two possible values of x.

3. Let $x=a^2-ab+1$ and $y=b^2+3ab-10$ for integers$a$ and b which satisfy $\begin{cases}1\le a<10\\10\le b<20\end{cases}$.
N is the <u>smallest</u> possible value of $x+y$ which will be a positive multiple of 11.
K different ordered pairs (a,b) produce this value of N.
Compute the ordered pair (N,K).

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Solutions: State Meet 2015

Round 2 Algebra I:

1. Clearly, a zero-product occurs if and only if $x = -\frac{2}{3}, -\frac{1}{6}$.
   Substituting in $12x - 5$, we get $-13, -7$.

2. Let the fraction be $\frac{x}{x+4}$. we have $\frac{x}{x+4} + \frac{x+4}{x} = \frac{2x^2+8x+4}{x(x+4)}$. Then we also have $1 - \frac{x}{x+4} = \frac{4}{x+4}$. The ratio of these $\frac{2x^2+8x+16}{4x}$ which equals 6.5 or $\frac{13}{2}$. Then $4x^2 + 16x + 32 = 52x$. Dividing by 4, we have $x^2 + 4x + 8 = 13x$, then $x^2 - 9x + 8 = 0$, whence $x =$ **1** or **8**.
   Note that, if difference were written as $\frac{x}{x+4} - 1 = \frac{-4}{x+4}$, the resulting quadratic would be $x^2 + 17x + 2 = 0$ which has no integer roots and is rejected.

3. $x + y = a^2 + 2ab + b^2 - 9 = (a+b)^2 - 9$
   We require that $x + y = (a+b)^2 - 9 = 11k$, for some positive integer k. Thus, $9 + 11k$ must be a perfect square. This is true for $k = 5 \Rightarrow 55 + 9 = 64 \Rightarrow a + b = 8$, but the minimum value of $a + b$ is 11. Continuing to add 11, we get the sequence 75, 86, 97, ….and the next perfect square is $196 = 11 \cdot 17 + 9 = 14^2 \Rightarrow x + y = 187$ $a + b = 14$ if $(a,b) = (1,13), (2,12), (3,11), (4,10)$ - 4 possible pairs. Thus, $(N,K) = \mathbf{(187,4)}$.

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Round 3 – Geometry

1. ______________________

2. ______________________

3. (______ , ______ , ______)

1. Regular Hexagon $ABCDEF$ is inscribed in a circle of radius 10 cm. What is the exact number of square centimeters in the area of ΔACD?

2. The ratio of twice the complement of an angle to its supplement is $\frac{4}{7}$. Determine the number of degrees in the measure of the angle.

3. Chords $\overline{AB}$ and $\overline{CD}$ each with integer lengths are drawn in circle of radius R with center at O. Each chord is an integer distance from the center O.The shorter chord $\overline{AB}$ is twice as far from O as the longer chord $\overline{CD}$, but the lengths of the chords are <u>not</u> in a 2 : 1 ratio.
If $CD = AB + 2$, compute the ordered triple $\left(AB, CD, R^2\right)$ for the <u>minimum</u> possible value of R.

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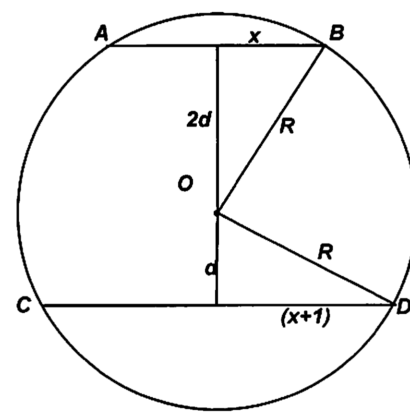
Solutions: State Meet 2015

Round 3 Geometry:

1. ΔACD is a 30-60-90 right triangle with right angle at C.
$\frac{1}{2}\cdot 10 \cdot 10\sqrt{3} = 50\sqrt{3}$

2. $\frac{2(90-x)}{180-x} = \frac{4}{7} \rightarrow 14(90-x) = 720-4x \Rightarrow x = 54$

3. Since distance is measured along a perpendicular and a radius drawn perpendicular to any chord in a circle bisects that chord, we let $AB = 2x$ and $CD = 2x+2$. Then:



$\begin{cases} x^2+4d^2 = R^2 \\ (x+1)^2+d^2 = R^2 \end{cases}$ Subtracting, $2x+1 = 3d^2 \Rightarrow d^2 = \frac{2x+1}{3}$

For d to be an integer, $\frac{2x+1}{3}$ must be a perfect square, so let

$2x+1 = 3k^2$, implying $x = \frac{3k^2-1}{2}$.

$k = 1 \Rightarrow x = 1$ which is rejected ($AB:CD = 2:4 = 1:2$)

$k = 2 \Rightarrow x = \frac{11}{2} \Rightarrow (AB, CD) = (11, 13)$ and $d = 2$

Computing R^2, we have $R^2 = \left(\frac{11}{2}\right)^2 + 4^2 = \left(\frac{13}{2}\right)^2 + 2^2 = \frac{185}{4}$

Thus, $(AB, CD, R^2) = \underline{\left(\mathbf{11, 13, \frac{185}{4}}\right)}$.

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Round 4 – Algebra 2

1. ________________________

2. ________________

3. ________________________

1. The roots of a quadratic equation $ax^2 + bx + c = 0$, are in the ratio 4:5. The product of the roots is the reciprocal of 20. Determine all possible coefficients of the *x*-term,given that the coefficients of the equation are relatively prime.

2. If $\log_{32}(66 + \log_4 b) = \frac{6}{5}$, compute the value of *b*.

3. The graphs of $y = |2x-3| - 7$ and $y = 13 - |2x - A|$ are partially coincident for <u>two</u>distinct values of A.

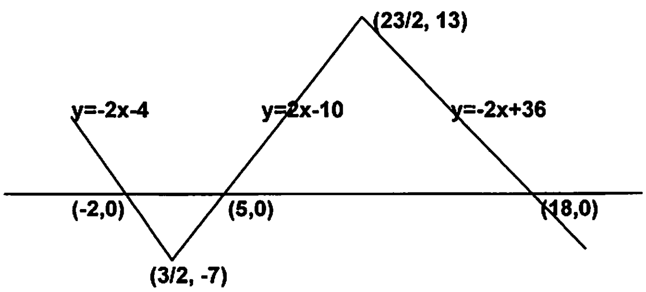
   Regardless of which value of A is used, the total area bounded by the two absolute value graphs and the *x*-axis is the same and equals B units^2. Compute B.

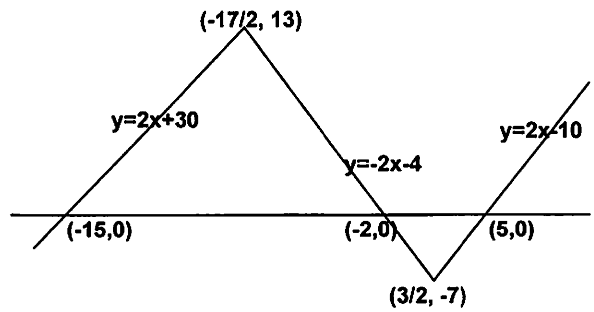
Solutions: State Meet 2015

Round 4 Algebra II

1. The roots are $\frac{1}{4}$ and $\frac{1}{5}$ or $-\frac{1}{4}$and $-\frac{1}{5}$. This leads to
$20x^2 \pm 9x + 1 = 0$.

2. From the definition of logs, $66 + \log_4 b = 64$
$\Rightarrow \log_4 b = -2 \Rightarrow b = \frac{1}{16}$





3. $y = |2x-3| - 7 \Leftrightarrow y = \begin{cases} 2x-10 & \text{for } x \geq \frac{3}{2} \\ -2x-4 & \text{for } x < \frac{3}{2} \end{cases}$

$y = 13 - |2x - A| \Leftrightarrow y = \begin{cases} -2x+(13+A) & \text{for } x \geq \frac{4}{2} \\ 2x+(13-A) & \text{for } x < \frac{4}{2} \end{cases}$

If $y = mx + b \Leftrightarrow y = 2x - 10$, then $13 - A = -10 \Rightarrow A = 23$

The required area is

$\frac{1}{2}(5+2)(7) + \frac{1}{2}(18-5)(13) = \frac{49+169}{2} = \underline{\mathbf{109}}$.

Note: If $y = mx + b \Leftrightarrow y = -2x - 4$, then $13 + A = -4 \Rightarrow A = -17$.

In either case, the enclosed area is the same.

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Round 5 – Analytic Geometry

1. ________

2. ________________________

3. (______ , ______ , ______)

1. A right triangle is such that the vertex opposite the hypotenuse is at the origin. The y-intercept of the hypotenuse is –3 times its x-intercept. The sum of these intercepts is 4. Compute the length of the median to the hypotenuse which has a negative slope.

2. An ellipse has the equation: $\frac{x^2}{9}+\frac{y^2}{25}=1$. Compute the area of an isosceles trapezoid whose height is 2, all of whose vertices are on the ellipse, and one of whose bases is the major axis of the ellipse.

3. In the ellipse with equation$9x^2+25y^2-18x+100y-116=0$, F is the leftmost focus, and P is the uppermost endpoint of the focal chord which does not pass through F.
The equation of line $\overleftrightarrow{PF}$ is $Ax+By+C=0$, where A, B and C are integers and $A>0$.
Compute the ordered triple (A,B,C) which has a minimum value of A.

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Solutions: State Meet 2015

Round 5 Analytic Geometry:

1. Let the intercepts be $(a, 0)$and $(0, -3a)$. Then $a + 3a = 4 \rightarrow a = -2$. The endpoints of the median are $(0, 0)$and $(1, -3)$. The length is $\sqrt{10}$. Solution $(-1, 3)$.

2. The major axis has length 10. The other base of the trapezoid passes through $(\pm 2, 0)$. From $\frac{4}{9} + \frac{y^2}{25} = 1$, we get $y = \pm\frac{5\sqrt{5}}{3}$. ∴the other base has length $\frac{10\sqrt{5}}{3}$. The area of the trapezoid is $\frac{1}{2} \cdot 2 \cdot \left(10 + \frac{10\sqrt{5}}{3}\right) = \frac{30+10\sqrt{5}}{3}$

3. Completing the square (twice) we have

$$9x^2 + 25y^2 - 18x + 100y - 116 = 0 \Leftrightarrow \frac{(x-1)^2}{25} + \frac{(y+2)^2}{9} = 1$$

A horizontal ellipse with major axis along $y = -2$, center at $(1,-2)$, $(a,b,c) = (5,3,4)$ with each focal chord having length $\frac{2b^2}{a} = \frac{18}{5}$.

Thus, $F(1-c,-2) = (-3,-2)$ and $P\left(1+c,-2+\frac{b^2}{a}\right) = \left(5,-\frac{1}{5}\right)$.

The slope of line $\overline{PF}$ is $\dfrac{-\frac{1}{5}+2}{5+3} = \dfrac{-1+10}{40} = \dfrac{9}{40}$

$$(y+2) = \frac{9}{40}(x+3) \Leftrightarrow 9x - 40y - 53 = 0 \Leftrightarrow (A,B,C) = \underline{\mathbf{(9,-40,-53)}}$$

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Round 6 – Trig and Complex Numbers

1. ______________________

2.(_ , _______)

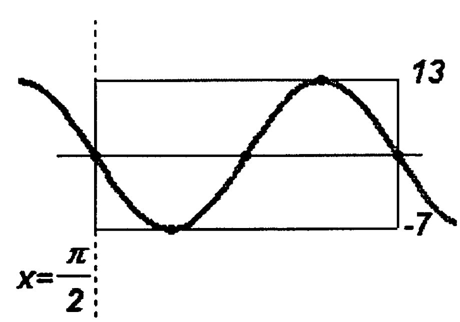
3. (____ , ____ , ____ , ____)

1. Evaluate: $\dfrac{\sin\left(\mathrm{Tan}^{-1}\frac{2}{3}\right)}{\cot\left(\mathrm{Cos}^{-1}\frac{3}{10}\right)}$

2. Find the ordered pair of integers (a, b) where $\dfrac{(2+i)^2}{a+bi} = -1 + 2i$.

3. A sine curve is defined by $y = f(x) = A\sin(Bx + C) + D$, where $A < 0$, $B > 0$, has these characteristics:
   - its maximum value is 13
   - its minimum value is –7,
   - its period is $\dfrac{4\pi}{5}$, and
   - its phase shift is $\frac{\pi}{2}$ (to the right) .

   Compute the ordered 4-tuple (A, B, C, D). A window into one period of this function is shown at the right.



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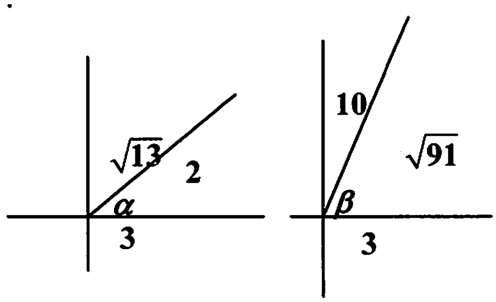
Solutions: State Meet 2015

Round 6Trigonometry and Complex Numbers:

1. Let $\alpha = \mathrm{Tan}^{-1}\left(\frac{2}{3}\right)$ and $\beta = \mathrm{Cos}^{-1}\left(\frac{3}{10}\right)$.
Since the arguments of the inverse trig functions are positive, the values (angles) are in quadrant 1.



$$\frac{\sin\alpha}{\cot\beta} = \frac{\frac{2}{\sqrt{13}}}{\frac{3}{\sqrt{91}}} = \frac{2\sqrt{91}}{3\sqrt{13}} = \underline{\mathbf{\frac{2\sqrt{7}}{3}}}$$

2. $a + bi = \frac{(2+i)^2}{-1+2i} = \frac{3+4i}{-1+2i} \cdot \frac{-1-2i}{-1-2i} = \frac{5-10i}{5} = 1 - 2i$

3. $(13-(-7)=20) \Rightarrow$ an amplitude of 10 and a centerline of $y = 3$.
Since this graph was reflected over the x-axis, we immediately have $A = -10$ and $D = -3$.
Since the period of $y = \sin x$ is 2π, the period of $y = f(x)$ is $\frac{2\pi}{|B|} = \frac{4\pi}{5}$.
Since $B > 0$, we have $B = \frac{5}{2}$. We are required to shift the "starting" period from $\left[0, \frac{4\pi}{5}\right]$ to $\left[0+\frac{\pi}{2}, \frac{4\pi}{5}+\frac{\pi}{2}\right] = \left[\frac{\pi}{2}, \frac{13\pi}{10}\right]$. Therefore, $0 \le \frac{5}{2}x + C \Rightarrow x \ge -C \cdot \frac{2}{5}$ and we have $\frac{-2C}{5} = \frac{\pi}{2} \Rightarrow C = -\frac{5\pi}{4}$. The equation of "flipped" sine function was $y = 10\sin\left[\frac{5}{2}\left(x - \frac{\pi}{2}\right)\right] + 3$, so the equation of the original sine function was $y = -10\sin\left[\frac{5}{2}\left(x - \frac{\pi}{2}\right)\right] - 3$ and

$(A,B,C,D) = \underline{\left(\mathbf{-10, \frac{5}{2}, -\frac{5\pi}{4}, -3}\right)}$.

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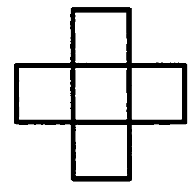
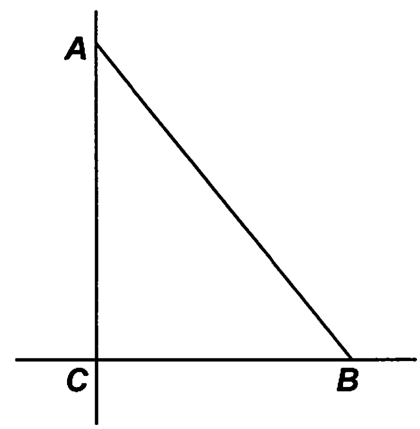
Team Round
Place answers on Team Round Answer Sheet

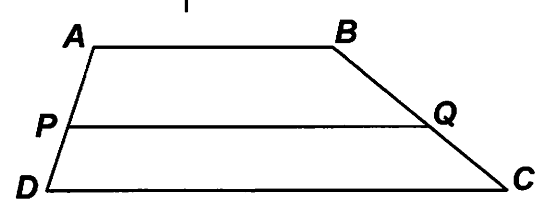
1. Five of the first six natural numbers are placed at random in the grid at the right, one number in each cell. Compute the probability that both 3-digit numbers in the grid are divisible by the 6^{th} number.



2. Given $A(0,24)$ and $B(h,0)$, where h is an integer and $0<h<100$. M lies on $\overline{AB}$ and N lies on $\overline{BC}$. For how many values of h do the medians $\overline{CM}$ and $\overline{AN}$ intersect in a lattice point, that is, a point both of whose coordinates are integers?



3. In trapezoid $ABCD$, $\overline{AB}\parallel\overline{PQ}\parallel\overline{DC}$, and $\dfrac{a(ABQP)}{a(PQCD)}=\dfrac{PQ}{DC}$, where $a(\ )$ denotes "area of".
If $PQ=h$ and $DC=k$, then $(AB)^2=\dfrac{h}{k}\cdot T$.
Find a simplified expression for T in terms of h and k.



4. The expansion of $\left(2\sqrt[3]{2}+3\sqrt[4]{3}+5\sqrt[5]{5}\right)^{18}$ produces k natural numbers. Compute k.

5. Let $a_0=1$ and $a_1=n$, where n is an integer greater than 1. For $j\ge 2$, subsequent values are defined by the recursive relation $a_j=\begin{cases}3\cdot a_{j-1}-1 & \text{for } j \text{ odd}\\ \dfrac{a_{j-1}}{3}+1 & \text{for } j \text{ even}\end{cases}$.
Consider $S_k=\sum_{i=0}^{i=k}a_i$ for $k\ge 2$. Let m be the <u>smallest</u> value of k for which S_k is an integer.
Let S be the <u>smallest</u> possible value of S_m. Determine the ordered pair (m,S).

6. Let $\theta=Cos^{-1}\left(\dfrac{3}{5}\right)+Cos^{-1}\left(-\dfrac{5}{13}\right)$. Compute $\sin 2\theta$.

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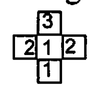
Solutions: State Meet 2015

Team:

1. There are 6! = 720 permutations of the first 6 natural numbers, with 120 starting with each of the 6 numbers. Let the first number denote the divisor.
Divisor 1: {2,3,4,5,6} – regardless of placement, divisibility by 1 is guaranteed __120__
Divisor 2: {1,3,4,5,6} – least significant digits must be even __12__
Divisor 3: {1,2,4,5,6}



If the center digit is 1, both the remaining horizontal pair and vertical pair must sum to 2 more than a multiple of 3 to insure that both 3-digit numbers are multiples of 3, call them $3j+2$ and $3k+2$. $(3j+2)+(3k+2)=3(j+k+1)+1 \Rightarrow$ sum of the remaining 4 digits must be 1 more than a multiple of 3, but this is not the case. The remaining 4 digits sum to $18-1=17$. Similar arguments exclude 2, 4 and 5 from being the center digit. Thus, the center digit must be 6.
The groupings must be $\{1,2,6\}$ and $\{4,5,6\}$. Each grouping gives rise to 2 different 3-digit numbers which may be used to fill horizontally, then vertically or vice versa. $2\cdot 2\cdot 2 \Rightarrow \underline{8}$

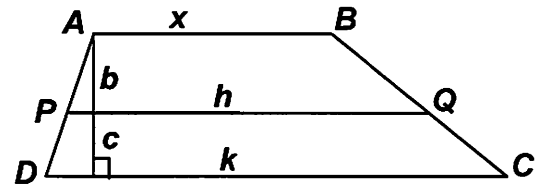
Divisor 4: {1,2,3,5,6} The center digit cannot be 2 or 6 since this would not allow two least significant digits both vertically and horizontally to form a multiple of 4. With a center digit of 1, there are 4 placements which produce multiples of 4.
This is also the case, if the center number is 3 or 5 . $4\cdot 3 \Rightarrow \underline{12}$

Divisor 5: {1,2,3,4,6} - impossible.
Divisor 6: {1,2,3,4,5}- must guarantee both divisibility by 2 and 3
Since the least significant digits must be even and the digit sums must be multiples of 3, the digit groupings must be {1,2,3} and {3,4,5} with 3 in the center position.
Thus, 132 and 534 placed vertically and horizontally or vice versa. __2__
The required probability is $\dfrac{(120+12+8+12+2)}{720}=\dfrac{154}{720}=\underline{\dfrac{77}{360}}$.

2. The equation of $\overleftrightarrow{AN}$ is $(y-24)=\dfrac{24}{-h/2}(x-0)\Rightarrow y=-\dfrac{48}{h}x+24$
The equation of $\overleftrightarrow{CM}$ is $y=\dfrac{12}{h/2}x=\dfrac{24}{h}x$.
Equating, $-\dfrac{48}{h}x+24=\dfrac{24}{h}x\Rightarrow\dfrac{72x}{h}=24\Rightarrow x=\dfrac{h}{3}$
Substituting, $y=\dfrac{24}{h}\cdot\dfrac{h}{3}=8$.
(Thus, the y-coordinate is always an integer. No surprise, if you are familiar with the fact that the medians of a triangle intersect at a point 2/3 the distance from each vertex.)
Since the point of intersection is $\left(\dfrac{h}{3},8\right)$, we have lattice points whenever h is a multiple of 3.
There are **33** such multiples, namely $3\cdot 1,\ldots,3\cdot 33$.

3.

$$\frac{a(ABQP)}{a(PQCD)}=\frac{\cancel{\frac{1}{2}}b(x+h)}{\cancel{\frac{1}{2}}c(h+k)}=\frac{h}{k}\Rightarrow \boxed{\frac{b}{c}=\frac{h(h+k)}{k(x+h)}}$$



$$a(ABQP)+a(PQCD)=a(ABCD)\Leftrightarrow$$

$$\frac{1}{2}b(x+h)+\frac{1}{2}c(h+k)=\frac{1}{2}(b+c)(x+k)$$

$$\Leftrightarrow \cancel{bx}+bh+ch+\cancel{ck}=\cancel{bx}+bk+cx+\cancel{ck}$$

$$\Leftrightarrow$$

$$bh-bk=cx-ch\Leftrightarrow b(h-k)=c(x-h)\Leftrightarrow$$

$\boxed{\frac{b}{c}=\frac{x-h}{h-k}}$. Equating the $\frac{b}{c}$ expressions and cross multiplying,

$$h\left(h^2-k^2\right)=k\left(x^2-h^2\right)$$

$$\Rightarrow x^2=\frac{h}{k}\left(h^2-k^2\right)+h^2=\frac{h}{k}\left(h^2+hk-k^2\right)\Rightarrow$$

$T=\underline{\boldsymbol{h^2+hk-k^2}}$.

4. Each term in the expansion will be of the form $k\left(2\sqrt[3]{2}\right)^a\left(3\sqrt[4]{3}\right)^b\left(5\sqrt[5]{5}\right)^c$, where $a+b+c=18$ and k is an integer coefficient***.

Thus, we must partition 18 into the sum of 3 integers a, b and c, where $3\,|\,a$, $4\,|\,b$ and $5\,|\,c$.

$a=18\Rightarrow(18,0,0)$

$a=15$ and $a=12$ fail

$a=9\Rightarrow(9,4,5)$

$a=6\Rightarrow(6,12,0)$

$a=3\Rightarrow(3,0,15)$

$a=0\Rightarrow(0,8,10)$

Thus, the expansion contains **5** natural numbers.

*** FYI:

k is called a multinomial coefficient, denoted $\binom{n}{a\ b\ c}$, where $a+b+c=n$. It is evaluated as

$\frac{n!}{a!\cdot b!\cdot c!}$. For example, $\binom{18}{9\ 4\ 5}=\frac{18!}{9!4!5!}=\frac{18\cdot 17\cdot \cancel{16\cdot 15}^{10}\cdot 14\cdot 13\cdot \cancel{12\cdot 10}}{\cancel{24}\cdot \cancel{120}}=306(1820)=556{,}920$

The number of terms in the simplified expansion $(a+b+c)^n$ is given by $\frac{(n+1)(n+2)}{2}$.

For example, $\left(2\sqrt[3]{2}+3\sqrt[4]{3}+5\sqrt[5]{5}\right)^{18}$ will have $\frac{19\cdot 20}{2}=190$ terms.

5. The first few terms of the sequence are:

$1,\ n,\ a_2 = \dfrac{n+3}{3},\ a_3 = n+2,\ a_4 = \dfrac{n+5}{3},\ a_5 = n+4,\ a_6 = \dfrac{n+7}{3}$

For the above sequence, we note that only even-indexed terms can generate a fractional value. Consider partial sums of even-indexed terms.

$a_2 + a_4 = \dfrac{2n+8}{3}$ will be fractional unless $n = 2,5,8,\ldots,\ 3c+2$ for non-negative values of c.

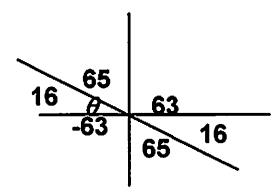
$a_2 + a_4 + a_6 = \dfrac{3n+15}{3} = n+5$ will be an integer for all values of n.

Thus, $m = 6$ is the minimum value of k for which S_k is always an integer. Specifically,

$$S_6 = \sum_{j=0}^{j=6} a_j = 1+n+(n+2)+(n+4)+(a_2+a_4+a_6) = (3n+7)+(n+5) = 4n+12$$

Since it was given that $n > 1$, the minimum value of S_6 is $4\cdot 2+12 = 20$ and $(m, S_m) = \underline{\mathbf{(6,20)}}$.

6. Let $\alpha = \text{Cos}^{-1}\left(\dfrac{3}{5}\right)$. [$\alpha$ must be in Quadrant 1.]



Let $\beta = \text{Cos}^{-1}\left(-\dfrac{5}{13}\right)$. [$\beta$ must be in Quadrant 2.]

Taking the tangent of both sides,

$$\tan\theta = \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\cdot\tan\beta} = \frac{\frac{4}{3}+\left(-\frac{12}{5}\right)}{1-\left(\frac{4}{3}\right)\left(-\frac{12}{5}\right)}\cdot\frac{15}{15} = \frac{20-36}{15+48} = \frac{-16}{63}$$

Note that $16^2+63^2 = 256+3969 = 4225 = 65^2$ must be in either quadrant 2 or 4.

$\sin 2\theta = 2\sin\theta\cos\theta = -2\left(\dfrac{16}{65}\right)\left(\dfrac{63}{65}\right) = \underline{-\dfrac{\mathbf{2016}}{\mathbf{4225}}}$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2015 – ANSWER SHEET

Round 1

1. 6
2. $\frac{23}{60}$
3. 6660

Round 2

1. $-13, -7$
2. 1, 8

3 (187, 4)

Round 3

1. $50\sqrt{3}$
2. 54
3. $\left(11, 13, \frac{185}{4}\right)$

Round 4

1. ± 9
2. $\frac{1}{16}$
3. 109

Round 5

1. $\sqrt{10}$
2. $\frac{30+10\sqrt{5}}{3}$
3. $(9, -40, -53)$

Round 6

1. $\frac{2\sqrt{7}}{3}$
2. $(1, -2)$
3. $\left(-10, \frac{5}{2}, -\frac{5\pi}{4}, -3\right)$

Team

1. $\frac{77}{360}$
2. 33
3. $h^2 + hk - k^2$
4. 5

5 $(6, 20)$

6. $-\frac{2016}{4225}$