This may not be the final draft of the 2018 Contest.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

STATE PLAYOFFS – 2018

Round 1 – Arithmetic and Number Theory

1. ____________________

2. ____________________

3. ____________________

1. "Reversible" primes are those that when their digits are reversed a prime number also results. Two particularpairsof distinct "reversible" primes are such that when added produce a two-digit multiple of 11. What is the sum of the 4 numbers in these two pairs?

2. Compute the probability that an integer chosen at random between 10 and 99, inclusive, will be divisible by 7 or 9?

3. Let n be an integer with $10 \le n \le 19$. Multiply$n$ by 5 and place the digits of $5n$to the left of the digits of n, producing the 4-digit number K. Compute the least possible sum of all the prime factors of K. (For the purpose of this question, the sum of all the prime factors of 24 is $2 + 2 + 2 + 3 = 9$)

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Round 2 – Algebra 1

1. ______________________

2. ______________________

3. ______________________

1. For positive integers x and y, compute the number of ordered pairs (x, y) satisfying $x^2 + y^2 + 2xy - 27x - 27y + 126 = 0$.

2. If a and b are distinct elements of $\left\{2^{-.5}, 2^{-1.5}, 2^{-2.5}, 2^{-3.5}, 2^{-4.5}, 2^{-5.5}, 2^{-6.5}\right\}$ and x and y are solutions to the system below, what is the <u>greatest</u> possible value of $x + y$?

$$\begin{aligned} ax + by &= \frac{1}{b} \\ bx + ay &= \frac{1}{a} \end{aligned}$$

3. Compute all values of k such that all the following equations have real solutions: $kx^2 + x + 1 = 0,\ x^2 + kx + 1 = 0,\ x^2 + x + k = 0$

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Round 3 – Geometry

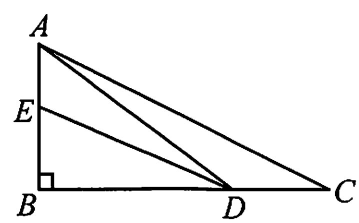
1. ______________________

2. ______________________

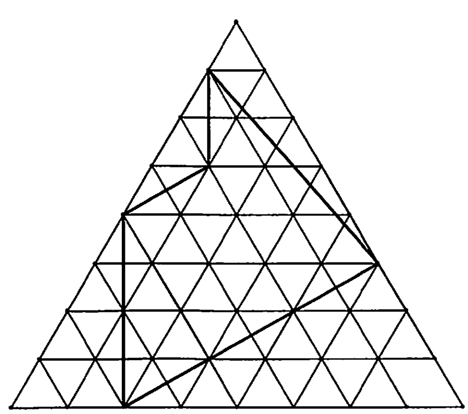
3. ______________________

1. A circle of radius 10 is circumscribed about a regular hexagon. A circle is then inscribed in the hexagon. Compute the exact number of square units in the area of the region between the inner circle and the hexagon.

2. In ΔABC, $AB = 6$, $BC = 12$, $m\angle ABC = 90°$, and the areas of EBD, AED, and ADC are equal. Compute the length of the altitude from E to $\overline{AD}$.



3. A large equilateral triangle is divided into quite a few equilateral triangles of area 1 as shown. Compute the area enclosed by the pentagon shown.



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STATE PLAYOFFS – 2018

Round 4 – Algebra 2

1. ______________________

2. ______________________

3. ______________________

1. Let $\lfloor a \rfloor$ = the greatest integer less than or equal to *a*. If $\dfrac{12}{\lfloor -\log A \rfloor} = 3$, then $m < A \le n$. Compute the value of $\dfrac{n}{m}$?

2. If $S = 1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + 9 + 10 - 11 - 12 + 13 + 14 - 15 - \ldots$, then let $S_1 = 1$, $S_2 = 1+2$ $S_2 = 1+2$, $S_3 = 1+2-3$, $S_4 = 1+2-3-4$, $S_5 = 1+2-3-4+5$, and so on. For which *n*does the sum S_n fall below –2018 for the first time?

3. The graphs of the polar equations $r = 2\cos\theta$ and $r = \dfrac{4}{2\cos\theta + \sin\theta}$ intersect in two points $P(2, 0)$ and $Q(s, \tan^{-1} k)$. Find the ordered pair (s, k).

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Round 5 – Analytic Geometry

1. ____________________

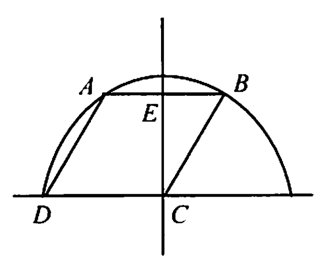
2. (______,_______,_______)

3. ____________________

1. Point P lies in the solution set of $3x+4y \le 10$ and point Q lies in the solution set of $3x+4y \ge 27$. Compute the minimum possible length of $\overline{PQ}$.

2. A parabola passes through the focal points and a y-intercept of $\frac{x^2}{25}+\frac{y^2}{9}=1$. If the equation of the parabola is written as $y = ax^2+bx+c$ for positive a, compute the ordered triple (a,b,c)?

3. $ABCD$ is a rhombus. Points A, B and D lie on a parabola whose equation is $f(x) = ax^2 + k$, C lies on the origin, E lies on the y-axis and is the midpoint of $\overline{AB}$. If $B = (z, f(z))$, compute the value of the product az.



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Round 6 – Trig and Complex Numbers

1. ______________________

2.______________________

3. ______________________

1. If $\sin a = 0.6$, with $90° < a < 180°$, compute the exact value of $\sin(2a) - \cos(2a)$ as a single fraction.

2. For angles in degrees, $(\cos 23 + \cos 67)(\sin 67 - \sin 23) = k\cos\theta$ where θ is an acute angle. Compute the ordered pair (k, θ)?

3. In the complex plane, the center of a square is at $5 + 3i$ and one vertex is at $7 + 8i$. Compute the sum of the coordinates of all four vertices.

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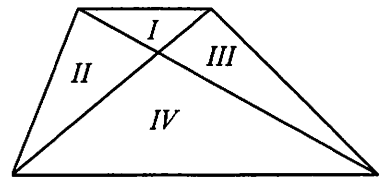
STATE PLAYOFFS – 2018

Team Round

Place answers on Team Round Answer Sheet.

1. In the trapezoid the Roman numerals *I, II, III*, and *IV* represent the areas of the four triangular regions that make up the trapezoid, with *IV* being the largest. If *I, II*, and *III* are integers and $IV = 256$, compute the largest possible area of the trapezoid.



2. How many integers from 5005 to 6006 inclusive have the property that the sum of the digits in the thousand's and ten's places equals the sum of the digits in the hundred's and one's places?

3. Let ${}_nB_k$ denote the number of ways to break a set of *n* elements into *k* disjoint non-empty sets. For example, ${}_3B_2 = 3$ since $\{a,b,c\}$ can be broken into 3 disjoint sets in the following ways: $\{a,b\} \cup \{c\}$, $\{a,c\} \cup \{b\}$, and $\{b,c\} \cup \{a\}$. Note that order doesn't matter. Compute ${}_7B_4$.

4. If $(a+bi)+(a+bi)^2+(a+bi)^3+\ldots=\frac{\sqrt{3}}{3}i$, what are the values of *a* and *b*? Write your answer as the ordered pair (a, b).

5. Compute the product of all the elements in the ordered triples (x,y,z) that satisfy the following system:

$$\begin{aligned} x + yz &= 11 \\ y + xz &= 11 \\ z + xy &= 11 \end{aligned}$$

6. At a restaurant each of 7 people ordered a different entrée. The newly hired server forgot who had ordered what and so he passed the entrees out at random. What is the probability that exactly 4 people did not receive what they had ordered?

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STATE PLAYOFFS – 2018 – ANSWER SHEET

Round 1

1. 132

2. $\frac{11}{45}$

3. 177

Round 2

1 25

2. 4096

3. $k \le -2$

Round 3

1. $150\sqrt{3} - 75\pi$

2. $\frac{12}{5}$

3. 30

Round 4

1. 10

2. 2020

3 $\left(\frac{4\sqrt{5}}{5}, \frac{1}{2}\right)$

Round 5

1 $\frac{17}{5}$

2. $\left(\frac{3}{16}, 0, -3\right)$

3. $-\frac{1}{\sqrt{3}}$

Round 6

1. $-\frac{31}{25}$

2. $(1,46)$

3. $20 + 12i$

Team

1. 961

2. 76

3. 350

4. $\left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$

5–3

6. $\frac{1}{16}$

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Solutions: State Meet 2018

Round 1 Arithmetic and Number Theory:

1. 13, 31 and 17, 71 are the two pair. The sum is 132.

2. There are 99 – 10 + 1 = 90 integers. There are $\frac{98-14}{7}+1=13$ multiples of 7 and $\frac{99-18}{9}+1=10$ multiples of 9. However, a common multiple, 63 is counted twice, so there are 13 + 10 – 1 = 22 integers divisible by 7 or 9. The probability that the integer will be divisible by 7 or 9 is $\frac{22}{90}=\boxed{\frac{11}{45}}$.

3. The 4-digit number, written as $(5n)(n) = (5n)100 + n = 501n = 3\cdot 167\cdot n$. Since both 3 and 167 are prime we need to find the least sum of the prime factors of n. Here's table showing n, its prime factors, and their sum:

n	10	11	12	13	14	15	16	17	18	19
prime factors	2, 5	11	2,2,3	13	2,7	3,5	4 2's	17	2,3,3,	19
sum	7	11	7	13	9	8	8	17	8	19

Thus, the least possible sum is 3 + 167 + 7 $=\boxed{177}$.

Round 2 Algebra I:

1. We have $(x+y)^2-26(x+y)+105=0$. This factors as $(x+y-5)(x+y-21)=0$. Thus, $x+y=5$ or $x+y=21$. There are 4 solutions to $x+y=5$, namely (1, 4) through (4, 1), and 20 solutions to $x+y=21$, namely (1, 20) through (20, 1). Answer: $\boxed{24}$.

2. Multiply the top equation by b and the bottom equation by a and subtract to eliminate the x term.

$$abx + b^2y = 1$$
$$abx + a^2y = 1$$

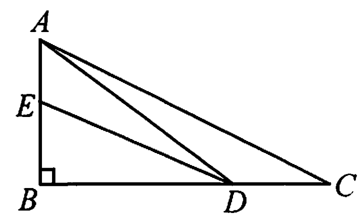
Then $(b^2-a^2)y=0\rightarrow y=0$. Since $y=0$ then $x=\frac{1}{ab}$. The largest possible sum of $x+y=0+\frac{1}{ab}=\frac{1}{ab}$. Choose the smallest possible values for a and b, namely $2^{-5.5}$ and $2^{-6.5}$ for which $\frac{1}{ab}=\frac{1}{2^{-5.5}\cdot 2^{-6.5}}=\frac{1}{2^{-12}}=2^{12}=\boxed{4096}$.

3. Solving we have $x = \frac{-1 \pm \sqrt{1-4k}}{2k}$, $\frac{-k \pm \sqrt{k^2-4}}{2}$, and $\frac{-1 \pm \sqrt{1-4k}}{2}$ respectively. From the middle expression we obtain $k^2 - 4 \geq 0 \to k \leq -2$ or $k \geq 2$. From the other two we have $1 - 4k \geq 0 \to k \leq \frac{1}{4}$. Thus for $\boxed{k \leq -2}$ we obtain real solutions for all three equations.

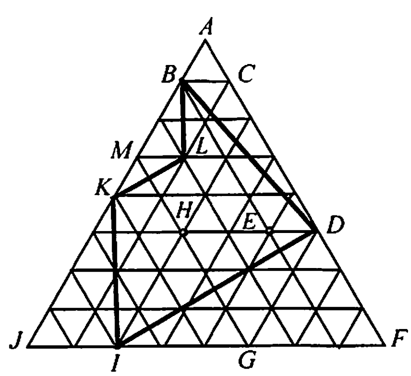
Round 3 <u>Geometry</u>:

1. The side of the hexagon is the same as the radius of the outer circle. The area of the hexagon is $\frac{3}{2} \cdot 100\sqrt{3} = 150\sqrt{3}$. The radius of the inner circle is $5\sqrt{3}$, so its area is 75π.

2 The area of ABC is 36 so the area of ADC is 12. Since the height of ADC is 6, $DC = 4$, making $BD = 8$. By the Pythagorean Theorem, $AD = \sqrt{6^2 + 8^2} = 10$. If h is the length of the altitude from E to $\overline{AD}$, then $\frac{1}{2} \cdot 10 \cdot h = 12 \to \boxed{h = \frac{12}{5}}$.



3. There are a total of 64 small equilateral triangles of area 1. A good strategy is to find the areas of the regions outside $BDIKL$ and then subtract from 64. ΔABC has an area of 1. Note that $\overline{BD}$ is a diagonal of parallelogram $BCDE$ formed by 8 equilateral triangles. So the area of ΔBCD is 4. ΔDGF has an area of 9. $\overline{DI}$ is the diagonal of parallelogram $DGIH$ which is formed by 18 triangles so the area of DGI is 9. The area of KJI is 8. The area of ΔKML is 1 and the area of ΔMLB is 2. Thus, the area of $BDIKL$ is $= 64 - (1+4+9+9+8+1+2) = 64 - 34 = \boxed{30}$.



Round 4 Algebra II

1. $\dfrac{12}{\lfloor -\log A \rfloor} = 3$ gives $4 = \lfloor -\log A \rfloor \to 4 \le -\log A < 5 \to -4 \ge \log A > 5 \to 10^{-4} \ge A > 10^{-5}$. Then $n = 10^{-4}$, $m = 10^{-5}$ and $\dfrac{n}{m} = \boxed{10}$.

2. Note that $S_1 = 1,\ S_2 = 3,\ S_3 = 0,\ S_4 = -4,\ S_5 = 1,\ S_6 = 7,\ S_7 = 0,\ S_8 = -8$, and so on. Note the pattern every 4 terms. If $n = 4k+1$, then the sum of the 4 terms is $(4k+1)+(4k+2) - (4k+3)+(4k+4)$ $= -4$. So the four terms ending with n equal to a multiple of 4 will add –4 to the sum. Thus, $S_4 = -4,\ S_8 = -8,\ S_{12} = -12$, and so on, giving $S_{2016} = -2016$. The next two terms give $S_{2017} = -2016 + 2017 = 1$ and $S_{2018} = 1 + 2018 = 2019$, $S_{2019} = 2019 - 2019 = 0$, and $S_{2020} = 0 - 2020 = -2020 < -2018$. Thus, $n = \boxed{2020}$.

3. $\cos\theta = \dfrac{4}{2\cos\theta + \sin\theta}$, so $\cos\theta(2\cos\theta + \sin\theta) = 2$, and $2\cos^2\theta + \sin\theta\cos\theta = 2$ Then $\sin\theta\cos\theta = 2(1 - \cos^2\theta) = 2\sin^2\theta$. Since $P(2, 0)$ is given, we can divide by $\sin\theta$ to get $\cos\theta = 2\sin\theta$, or $\tan\theta = \dfrac{1}{2}$. So $k = \dfrac{1}{2}$, and $\cos x = \dfrac{2}{\sqrt{5}}$. Substituting into $r = 2\cos\theta$ we get

$$s = 2\left(\frac{2}{\sqrt{5}}\right) = \frac{4\sqrt{5}}{5}$$

Alternate solution:

The equation of the circle is $x^2 + y^2 = 2x$ and the equation of the line is $2x + y = 4$. Substituting we get $x^2 + (4 - 2x)^2 = 2x$ which becomes $5x^2 - 18x + 16 = 0$. Since we know that one solution is $x = 2$, we get $(x - 2)(5x - 8) = 0$. This gives $Q\left(\frac{8}{5}, y\right)$ and $y = 4 - \frac{16}{5} = \frac{4}{5}$. Then $\tan\theta = \frac{4}{5} / \frac{8}{5} = \frac{1}{2} = k$,

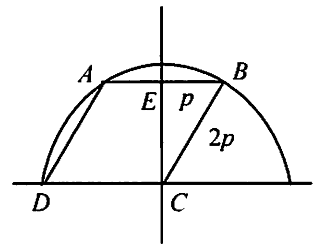
$s^2 = \left(\frac{8}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{80}{25}$, and $s = \frac{4\sqrt{5}}{5}$.

Round 5 Analytic Geometry:

1. The minimum length will be the distance from $3x + 4y = 10$ and $3x + 4y = 27$. Let $P = (2,1)$ where P lies on $3x + 4y = 10$. Then the distance from P to $3x + 4y = 27$ is $\dfrac{|3 \cdot 2 + 4 \cdot 1 - 27|}{\sqrt{3^2 + 4^2}} = \dfrac{|-17|}{5} = \boxed{\dfrac{17}{5}}$.

2. Since in an ellipse, $a^2 = b^2 + c^2$ we have $25 = 9 + c^2$ so the focal points are $(\pm 4, 0)$. This gives an equation of the form $y = k(x^2 - 4)$. Since $a > 0$ then $k > 0$ and the y-intercept that we use is (0, –3). Then $-3 = k(0-16) \to k = \frac{3}{16}$ and the equation of the parabola is $y = \frac{3}{16}(x^2 - 16) = \frac{3}{16}x^2 + 0x - 3$. Answer: $\boxed{\left(\frac{3}{16}, 0, -3\right)}$.

3. Let $BC = 2p$, then since E is the midpoint of $\overline{AB}$, $EB = p$, and $EC = p\sqrt{3}$. The coordinates of B are therefore $(p, p\sqrt{3})$. Thus $p\sqrt{3} = ap^2 + k$. Since $D = (-2p, 0)$ also lies on the parabola, $0 = a(-2p)^2 + k$. Subtracting the first from the second cancels the k's and gives $-p\sqrt{3} = 3ap^2$. Thus $ap = -\frac{1}{\sqrt{3}}$, so $\boxed{az = -\frac{1}{\sqrt{3}}}$.



Round 6<u>Trigonometry and Complex Numbers</u>:

1. .$\sin a = \frac{7}{10} \to \cos a = \frac{\sqrt{51}}{10}$. Using $\sin 2a = 2\sin a \cos a$ gives $\sin 2a = \frac{7\sqrt{51}}{50}$. Using $\cos 2a = 2\cos^2 a - 1$ gives $\cos 2a = \frac{1}{50}$. Subtracting gives $\frac{7\sqrt{51} - 1}{50}$

2. $(\cos 23 + \cos 67)(\sin 67 - \sin 23) = \cos 23 \sin 67 - \cos 23 \sin 23 + \cos 67 \sin 67 - \cos 67 \sin 23$. Note that since $\cos 23 = \sin 67$ and $\sin 23 = \cos 67$, then the middle two terms cancel leaving $\sin 67 \cos 23 - \cos 67 \sin 23 = \sin(67 - 23) = \sin 44 = \cos 46$. Answer: $\boxed{(1, 46)}$

3. Translate the center back to the origin by subtracting 5 + 3i. Do the same for the vertex, thereby obtaining $2 + 5i$. Multiplying $2 + 5i$ by i, i^2, and i^3 will rotate $2 + 5i$, by 90°, 180·, and then 270°, resulting in $-5 + 2i$, $-2 - 5i$, and $5 - 2i$. These will be the vertices of a congruent square centered at the origin. Add $5 + 3i$ to each of these to obtain the coordinates of the original square. Then add those 4 coordinates. Note that the sum of the coordinates of the square centered at the origin is 0, so the answer to the problem is $4(5 + 3i) = \boxed{20 + 12i}$.

Team:

1. It helps to know that the areas form a geometric sequence and that *II* =*III*. Thus, we seek integers such that $(II)^2 = 256 \cdot I$. Note that *I* must be a perfect square. Here are a couple of candidates for the areas and their sums:

1-16-16-256	289	4-32-32-256	324
9-48-48-256	361	16-64-64-256	400

Since *I* must be an integer and a perfect square, it is clear that the largest area will occur when $I = 15^2$, giving 225-240-240-256 and a sum of $\boxed{961}$.

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2. If a number is divisible by 11 the sum of the digits in the odd places of the number minus the sum of the digits in the even places is a multiple of 11. The number of multiples of 11 in the interval [5005, 6006] is $\frac{6006 - 5005}{11} + 1 = 92$. Many of those have the property that the difference between the sum of the digits in the first and third places and the second and fourth places is 0, but some don't. Those have a difference of 11 or –11 and we must eliminate those. If the difference is 11 we have the following:

11–0	5060
12–1	5170, 5071
13–2	5280, 5082, 5181
14–3	5390 5093, 5291, 5192

If the difference is –11 we have the following:

5–16	5907, 5709, 5808
6–17	5918, 5819
7–18	5929

Thus, we must reject 10 + 6 numbers giving 92 – 16 =$\boxed{76}$ numbers with the desired property.

3. Possibilities are 4-1-1-1, 3-2-1-1, or 2-2-2-1. If 4-1-1-1 we have ${}_7C_4 = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ ways to pick elements for the set containing 4 elements. The other three sets consisting of one element are determined and so there is no need to count them. If 3-2-1-1 we have ${}_7C_3 \cdot {}_4C_2 = 35 \cdot \frac{4 \cdot 3}{2 \cdot 1} = 210$ ways to pick the elements for a set of three elements and two elements for the set of two elements. The two sets of one element are determined so there is no need to count them. If we have 2-2-2-1, then we have ${}_7C_2 \cdot {}_5C_2 \cdot {}_3C_2 = \frac{7 \cdot 6}{2 \cdot 1} \cdot \frac{5 \cdot 4}{2 \cdot 1} \cdot 3 = 630$.

However, this counts the ordering of the three sets of 2 so we must divide 630 by 3!, obtaining 105. Total: 35 + 210 + 105 = $\boxed{350}$.

4. Using the formula for the sum of an infinite series we have $\dfrac{a+bi}{1-(a+bi)}=\dfrac{i\sqrt{3}}{3}$ giving $3a+3bi=b\sqrt{3}+\sqrt{3}(1-a)i$. Thus, $3a=b\sqrt{3}\rightarrow b=a\sqrt{3}$ and $3b=\sqrt{3}(1-a)\rightarrow b\sqrt{3}=1-a$. Then $(a\sqrt{3})\sqrt{3}=1-a$ so $a=\dfrac{1}{4}$ and $b=\dfrac{\sqrt{3}}{4}$.

Answer: $\boxed{\left(\dfrac{1}{4},\dfrac{\sqrt{3}}{4}\right)}$.

5. From $yz=11-x$ and $xz=11-y$ we obtain $\dfrac{yz}{xz}=\dfrac{y}{x}=\dfrac{11-x}{11-y}$ which equals $x^2-y^2=11(x-y)\rightarrow(x-y)((x+y)-11)=0$. Thus, $x=y$ or $x+y=11$. By the symmetry involved, if $x=y$ then $x=y=z$, and we have the equation $x^2+x-11=0$ and the solutions are $x=\dfrac{-1\pm3\sqrt{5}}{2}$. This gives two ordered triples, one whose terms are all $\dfrac{-1+3\sqrt{5}}{2}$ or $\dfrac{-1-3\sqrt{5}}{2}$. The sum of the 6 elements in these two ordered triples is $6\left(-\dfrac{1}{2}\right)=-3$ since the radicals cancel. Also by the symmetry involved, if $x+y=11$ then $x+z=11$ giving $y=11-x$ and $z=11-x$. Substituting gives $x+(11-x)^2=11$ or $x^2-21x+110=0\rightarrow(x-11)(x-10)=0\rightarrow x=11$ or 10. If $x=11$ then $y=z=0$, but $(11,0,0)$ can't be a solution to all three equations so we reject it along with (0, 11, 0) and
(0, 0, 11). If $x=10$ then $y=z=1$ and (10, 1, 1) does solve all three equations. Likewise (1, 10, 1) and (1, 1, 10) also solve all three equations. The sum of the elements of these three ordered triples is 36. The final sum is $\boxed{-3}$.

6. Let $D(x)$ stand for the number of derangements of x people, i.e., the number of ways x people can be arranged so that no one is in their original place. There are 7! possible ways to distribute the orders at random, $\dbinom{7}{3}$ ways to select a group of 3 people who got their order, and $D(4)$ ways for the other 4 to fail to receive the correct order. To calculate $D(4)$ look at all permutations of 1 2 3 4. If the permutation starts with 1 we discount it since it is not a derangement. If the permutation starts with 2, then there are 3 permutations in which none of 1, 3, and 4 is in their original location. The same holds true if the permutation starts with 3 or 4. Thus, $D(4)=9$. Then, the probability that exactly 4 don't receive their order is $\dfrac{{}_7C_3\cdot D(4)}{7!}=\dfrac{\frac{7!}{3!\cdot4!}\cdot9}{7!}=\dfrac{9}{3!\cdot4!}=\boxed{\dfrac{1}{16}}$.