Forty-Ninth Annual

MAML State Championship Math Meet

Friday, April 8, 2022

Conducted by

THE MASSACHUSETTS ASSOCIATION
OF MATHEMATICS LEAGUES

CONTEST

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

FORTY-NINTH STATE PLAYOFFS

Friday, April 8, 2022

Round 1. Arithmetic and Number Theory

1. ______________________

2. ______________________

3. ______________________

1. What is the smallest counting number which is divisible by all counting numbers less than 17?

2. In the multiplication below, different digits from 0 to 9 have been replaced by different letters. Find the five-digit product $ACDBC$.

$$\begin{array}{ccccc} & & A & B & C \\ & & A & B & C \\ \hline & & D & B & C \\ & B & C & E & \\ A & B & C & & \\ \hline A & C & D & B & C \end{array}$$

3. Reduce $\dfrac{39091}{69161}$ to lowest terms.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

FORTY-NINTH STATE PLAYOFFS

Friday, April 8, 2022

Round 2. Algebra I

1. ______________________

2. ______________________

3. ______________________

1. A, B, and C are thermometers with different scales. When A reads 12° and 36°, B reads 13° and 29°, respectively. When B reads 20° and 32°, C reads 57° and 84°, respectively. If the temperature drops 18° using A's scale, how many degrees does it drop using C's scale?

2. In Professor Tuffguy's mathematics class, 36 students took the final exam. If the average passing grade was 78, the average failing grade was 60, the class average was 71, and the passing grade was 70, how many of these 36 students passed the final?

3. Bill can complete a job alone in 4 days. When his son "helps," it takes 5 days for the two of them to complete the job. If Carl can complete the same job alone in 10 days, how many days would it take Carl and Bill's son together to complete the job? (Assume that Bill and Carl work — and that Bill's son interferes — at a constant rate.)

Round 3. Geometry

1. ______________________________

2. ______________________________

3. ______________________________

1. In triangle ABC, $BC = 6$, $AB = 9$, and $m\angle B > m\angle C$. List all possible integral values of AC.

2. An isosceles trapezoid is circumscribed about a circle. The bases of the trapezoid are 6 and 8. Find the radius of the circle.

3. A square is inscribed in a 90° sector of a circle so that two of its vertices are on the circle and the other two vertices are on the pair of perpendicular radii. If a side of the square is 6 meters long and the area of the region bounded by the entire circle is $k\pi$ square meters, find k.

Round 4. Algebra II

1. ______________________________

2. ______________________________

3. ______________________________

1. If $f(x) = \sqrt{x-3}$ and $g(x) = x^3 - 1$, find $f(g(2)) + g(f(12))$.

2. Solve for x:
$$4^x - 2^{x+3} + 12 = 0.$$

3. Let a_1, a_2, a_3, $\ldots$ be terms of an arithmetic sequence. If $a_4 + a_{10} = 12$ and $(a_4)(a_{10}) = 35$, then determine all possible sums of the first 15 terms.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

FORTY-NINTH STATE PLAYOFFS

Friday, April 8, 2022

Round 5. Analytic Geometry

1. ______________________

2. ______________________

3. ______________________

1. Find the shortest possible distance from the point $(-4, 6)$ to a point on the circle

$$(x-1)^2 + (y+2)^2 = 25.$$

2. Find the value of k so that the graph of $x^2 + 2xy + y^2 - 2x - 2y = k$ will be two parallel lines, one with y-intercept of 5.

3. Determine the value of a so that the curve $y = x^2$ is tangent to the curve $y = a(x-3)^2 + 6$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

FORTY-NINTH STATE PLAYOFFS

Friday, April 8, 2022

Round 6. Trigonometry and Complex Numbers

1. ______________________

2. ______________________

3. ______________________

1. Simplify as much as possible:

$$\frac{1-\cos\theta+\cos 2\theta}{\sin 2\theta-\sin\theta}$$

2. If $\tan 35^\circ = w$, express $\tan 10^\circ$ in terms of w.

3. Let z denote a complex number and let $\overline{z}$ be its conjugate. Find all values of z for which $(z)(\overline{z}) = 5$ and $z^2 + \overline{z^2} = 6$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

FORTY-NINTH STATE PLAYOFFS

Friday, April 8, 2022

Team Round.

1. The fraction 59/120 can be written as the sum of unit fractions in many ways. For example, $59/120 = 1/4 + 1/8 + 1/15 + 1/20$, where the sum of the denominators is 47. Determine the smallest possible sum of the denominators in such a sum of unit fractions.

2. Find the ordered triple (x, y, z) of integers for which
$$\left(\frac{25}{8}\right)^x \left(\frac{6}{25}\right)^y \left(\frac{16}{15}\right)^z = 2.$$

3. In $\triangle ABC$, D is the midpoint of $\overline{BC}$, E is the midpoint of $\overline{AD}$, F is the midpoint of $\overline{BE}$, and G is the midpoint of $\overline{CF}$. Find the ratio of the area of $\triangle EFG$ to the area of $\triangle ABC$.

4. Determine all real solutions to
$$x\sqrt{(x-1)^2} - 7x + 8 = 0.$$

5. The points of intersection of the graphs of
$$5x^2 - 3y^2 - 10x - 18y - 37 = 0$$
and
$$3x^2 - 5y^2 - 6x - 30y - 27 = 0$$
are the vertices of a convex quadrilateral. Find the area of this quadrilateral.

6. Find the radian values of x, $0 \le x < 2\pi$, such that
$$\sin 3x - \sin 2x - \sin x = 0.$$

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Conducted by

The Massachusetts Association
of Mathematics Leagues

ANSWER KEY

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

FORTY-NINTH STATE PLAYOFFS

Friday, April 8, 2022

ANSWER KEY

Arithmetic and Number Theory

1. 720720
2. 15625
3. 13/23

Algebra I

1. 27
2. 22
3. 20 or 20 days

Geometry

1. 10, 11, 12, 13, 14 (in any order)
2. $2\sqrt{3}$
3. 90

Algebra II

1. 28
2. 1 and $\log_2 6$ (in either order). $1+\log_2 3$ is also acceptable for the second answer.
3. 85, 95

Analytic Geometry

1. $\sqrt{89}-5$
2. 15
3. -2

Trigonometry and Complex Numbers

1. $\cot\theta$
2. $\frac{1-w}{1+w}$
3. $2+i, 2-i, -2+i, -2-i$ (in any order)

Team Round

1. 19
2. (3,2,2)
3. 1/8
4. $4+2\sqrt{2}$, $4-2\sqrt{2}$, $-3-\sqrt{17}$ (in any order)
5. 30
6. $0, \frac{2\pi}{3}, \frac{4\pi}{3}, \pi$ (in any order)

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OF MATHEMATICS LEAGUES

SOLUTIONS

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

FORTY-NINTH STATE PLAYOFFS

Friday, April 8, 2022

SOLUTIONS

Round 1 — Arithmetic and Number Theory

1. What is the smallest counting number which is divisible by all counting numbers less than 17?

   Answer: $\boxed{720720}$

   Solution: (This question is MAML 1975, R1Q1.) For a number to be divisible by all counting numbers between 1 and 16, its prime factorization must contain the prime factorization of each of these numbers. The greatest number of twos is within 16; the greatest number of threes is within 9; and each other prime factor less than 17 occurs at most once in any number from one to sixteen.

   So our answer equals $2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1$. Make the problem easier by grouping terms in helpful ways. $(2^3 \cdot 3^2) \cdot (2^1 \cdot 5^1) = (8 \cdot 9) \cdot 10 = 720$, and $7 \cdot 11 \cdot 13$ "famously" equals 1001. So the answer is $720 \cdot 1001 = \boxed{720720.}$

2. In the multiplication below, different digits from 0 to 9 have been replaced by different letters. Find the five-digit product $ACDBC$.

$$\begin{array}{ccccc} & & A & B & C \\ & & A & B & C \\ \hline & & D & B & C \\ & B & C & E & \\ A & B & C & & \\ \hline A & C & D & B & C \end{array}$$

   Answer: $\boxed{15625}$

   Solution: (This question is MAML 1972, R1Q2.) Notice that the last digit of C^2 is C. Four digits C could do this: 0, 1, 5, 6. C cannot be 0 because $ABC \cdot 0 = 000 \neq DBC$, and C cannot be 1 because $ABC \cdot 1 = ABC \neq DBC$. This leaves 5 and 6. But if C is 6, we have $AB6 \cdot 6 = DB6$, and subtracting $AB6$ from both sides we get $AB6 \cdot 5 = DB6 - AB6 = (D - A) \cdot 100$. Dividing both sides by 5 gives $AB6 = (D - A) \cdot 20$. The left hand number ends in 6, but the right hand number ends in 0, a contradiction. So C must be 5.

   Next consider A. Because $ABC \cdot A = ABC$ (the third line of the multiplication), A is 1.

   We now have $1B5 \cdot 5 = DB5$. When we multiply $5 \cdot 5$, a 2 is carried, so $5 \cdot B + 2$ must have a one's digit of B. Since B is an integer, $5 \cdot B + 2$ can end only in 2 or 7, so B can only be 2 or 7. If B is 7, we have $175 \cdot 5$ which equals 875. But the second line of multiplication would be $175 \cdot 7$, which has more than three digits.

   So our original number ABC must be 125.

$$\begin{array}{ccccc} & & 1 & 2 & 5 \\ & & 1 & 2 & 5 \\ \hline & & 6 & 2 & 5 \\ & 2 & 5 & 0 & \\ 1 & 2 & 5 & & \\ \hline 1 & 5 & 6 & 2 & 5 \end{array}$$

which matches the pattern.

3. Reduce $\dfrac{39091}{69161}$ to lowest terms.

   Answer: $\boxed{13/23}$

   Solution: (This question is MAML 1980, R1Q3.) Notice that both 39 and 91 share a common factor 13, so $39091 = 39 \cdot 1000 + 91$ is a multiple of 13. Dividing, $39091 = 13 \cdot 3007$. Conveniently, $69161 \div 3007 = 23$, leading to the final answer.

   Another way to do this problem uses the Euclidean Algorithm, which at its simplest states that the GCD of any two positive integers equals the GCD of the smaller integer and the difference of the two integers. $(69161, 39091) = (\underline{30070}, 39091) = (30070, \underline{9021}) = (\underline{21049}, 9021) = (\underline{12028}, 9021) = (\underline{3007}, 9021) = (3007, \underline{6014}) = (3007, \underline{3007})$. Obviously the GCD of 3007 and itself is 3007, so this factor can be divided from both the numerator and denominator, leading to the answer.

Round 2 — Algebra I

1. A, B, and C are thermometers with different scales. When A reads 12° and 36°, B reads 13° and 29°, respectively. When B reads 20° and 32°, C reads 57° and 84°, respectively. If the temperature drops 18° using A's scale, how many degrees does it drop using C's scale?

   Answer: $\boxed{27}$

   Solution: (This question is MAML 1973, R1Q3.) From the given information, a 24° increase on thermometer A equals a 16° increase in thermometer B, a $3:2$ (or $6:4$) ratio. Similarly, a 12° increase on thermometer B equals a 27° increase in thermometer B, a $4:9$ ratio.

   Connecting these three ratios, we see that $A:B:C = 6:4:9$, so $A:C = 6:9 = 2:3$. Therefore an 18° drop on thermometer A will correspond to a $\boxed{27^\circ}$ drop on thermometer C.

2. In Professor Tuffguy's mathematics class, 36 students took the final exam. If the average passing grade was 78, the average failing grade was 60, the class average was 71, and the passing grade was 70, how many of these 36 students passed the final?

   Answer: $\boxed{22}$

   Solution: (This question is MAML 1977, R1Q2.) Assume that p students passed the final; then $36-p$ failed. This gives us

$$78p + 60(36-p) = 71 \cdot 36$$
$$78p + 60 \cdot 36 - 60p = 71 \cdot 36$$
$$18p = 11 \cdot 36$$
$$p = \boxed{22}$$

3. Bill can complete a job alone in 4 days. When his son "helps," it takes 5 days for the two of them to complete the job. If Carl can complete the same job alone in 10 days, how many days would it take Carl and Bill's son together to complete the job? (Assume that Bill and Carl work — and that Bill's son interferes — at a constant rate.)

   Answer: $\boxed{\text{20 or 20 days}}$

   Solution: (This question is MAML 1979, R1Q3.) If Bill can complete the job alone in 4 days, he can complete 1/4 of the job in one day. If Bill and his son can complete the job together in 5 days, they complete 1/5 of the job in one day. Therefore Bill's son's "completes" $\frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$ of the job per day.

   Carl completes 1/10 of the job in one day. So if Carl and Bill's son work on the job together, they can complete $\frac{1}{10} - \frac{1}{20} = \frac{1}{20}$ of the job one day, so it takes $\boxed{20}$ days to complete the job.

Round 3 — Geometry

1. In triangle ABC, $BC = 6$, $AB = 9$, and $m\angle B > m\angle C$. List all possible integral values of AC.

   Answer: $\boxed{10,\ 11,\ 12,\ 13,\ 14 \text{ (in any order)}}$

   Solution: (This question is MAML 1984, R3Q1.) In a single triangle, a bigger side is always opposite a bigger angle, and conversely. Since $m\angle B > m\angle C$, we know that $AC > AB$ or $AC > 9$.

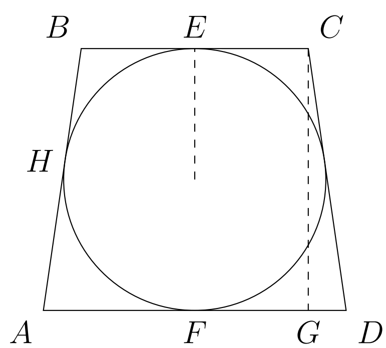
   By the Triangle Inequality, we know $AC < AB + BC$, so $AC < 9 + 6$.

   Therefore $9 < AC < 15$. The integers between 9 and 15 are $\boxed{10,\ 11,\ 12,\ 13,\ 14.}$

2. An isosceles trapezoid is circumscribed about a circle. The bases of the trapezoid are 6 and 8. Find the radius of the circle.

   Answer: $\boxed{2\sqrt{3}}$

   Solution: (This question is MAML 1973, R3Q3.) Drawing a diagram:

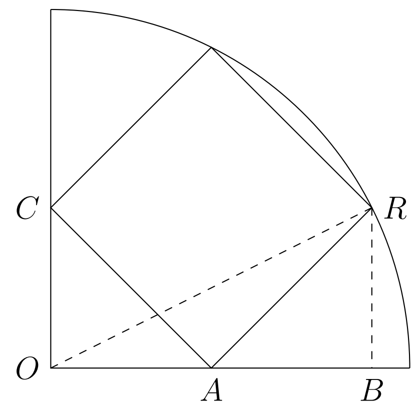


   By symmetry, $BE = CE = 3$ and $AF = DF = 4$. The lengths of both tangent segments to a circle from the same external point are equal, so $BH = BE = 3$ and $AH = AF = 4$. Thus, both legs of the trapezoid are $3 + 4 = 7$. Using the Pythagorean theorem on $\triangle CGD$ (with $CD = 7$ and $GD = 4 - 3 = 1$), the height of the trapezoid is $\sqrt{7^2 - 1^2} = \sqrt{48} = 4\sqrt{3}$. This equals the diameter of the inscribed circle, so the radius is $\boxed{r = 2\sqrt{3}.}$

3. A square is inscribed in a 90° sector of a circle so that two of its vertices are on the circle and the other two vertices are on the pair of perpendicular radii. If a side of the square is 6 meters long and the area of the region bounded by the entire circle is $k\pi$ square meters, find k.

Answer: $\boxed{90}$

Solution: (This question is MAML 1976, R3Q3.) Drawing a diagram:



The area of the entire circle is πr^2; so $k = r^2$ and our immediate goal is to find r, the radius of the circle. In the diagram $r = OR$.

There are two isosceles right triangles $\triangle AOC$ and $\triangle ABR$ with hypotenuses 6; so $OA = AB = BR = \frac{6}{\sqrt{2}} = 3\sqrt{2}$. Thus $OB = 6\sqrt{2}$ and $BR = 3\sqrt{2}$; the Pythagoran theorem gives $r = OR = \sqrt{90}$, so $\boxed{k = 90.}$

Round 4 — Algebra II

1. If $f(x) = \sqrt{x-3}$ and $g(x) = x^3 - 1$, find $f(g(2)) + g(f(12))$.

   Answer: $\boxed{28}$

   Solution: (This question is MAML 1994, R4Q1.)

$$f(g(2)) + g(f(12))$$

$$f(2^3 - 1) + g(\sqrt{12-3})$$

$$f(7) + g(3)$$

$$\sqrt{7-3} + (3^3 - 1)$$

$$2 + 26$$

$$\boxed{28}$$

2. Solve for x:

$$4^x - 2^{x+3} + 12 = 0.$$

   Answer: $\boxed{1 \text{ and } \log_2 6 \text{ (in either order). } 1 + \log_2 3 \text{ is also acceptable for the second answer.}}$

   Solution: (This question is MAML 1974, R4Q3.) We have

$$4^x - 2^{x+3} + 12 = 0$$

$$4^x - 8 \cdot 2^x + 12 = 0$$

$$(2^x)^2 - 8 \cdot 2^x + 12 = 0$$

$$(2^x - 6)(2^x - 2) = 0$$

   So either

$$2^x = 6 \qquad \text{or} \qquad 2^x = 2.$$

   In the former case, we have $\boxed{x = \log_2 6}$; in the latter case, we have $\boxed{x = 1.}$ Note that $\log_2 6 = \log_2(2 \cdot 3) = \log_2 2 + \log_2 3 = 1 + \log_2 3$, the alternate answer.

3. Let a_1, a_2, a_3, … be terms of an arithmetic sequence. If $a_4 + a_{10} = 12$ and $(a_4)(a_{10}) = 35$, then determine all possible sums of the first 15 terms.

   Answer: $\boxed{85,\ 95}$

   Solution: (This question is MAML 1988, R4Q3.) We have $a_4 a_{10} = 35$ and $a_4 + a_{10} = 12$. Substituting and solving, we have $(a_4, a_{10}) = (5, 7)$ or $(a_4, a_{10}) = (7, 5)$.

   If $a_4 = 5$ and $a_{10} = 7$, then $a_7 = 6$ and $a_1 = 4$. So the common difference between terms is $1/3$, and the fifteenth term is $4 + 14/3 = 26/3$. Then the sum is $\frac{(4+26/3)(15)}{2} = \frac{60+130}{2} = \boxed{95.}$

   If $a_4 = 7$ and $a_{10} = 5$, then $a_7 = 6$ and $a_1 = 8$. So the common difference between terms is $-1/3$, and the fifteenth term is $8 - 14/3 = 10/3$. Then the sum is $\frac{(8+10/3)(15)}{2} = \frac{120+50}{2} = \boxed{85.}$

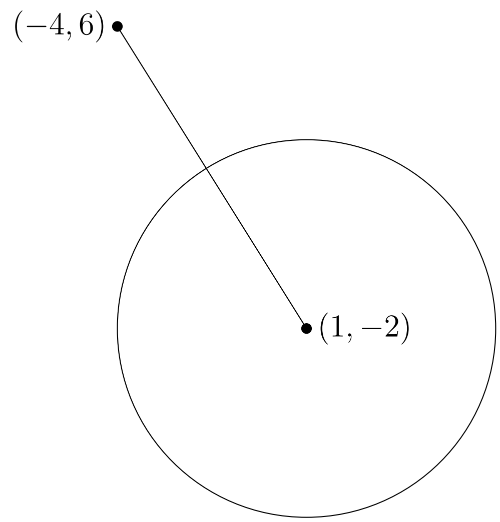
Round 5 — Analytic Geometry

1. Find the shortest possible distance from the point $(-4,6)$ to a point on the circle

$$(x-1)^2+(y+2)^2=25.$$

Answer: $\boxed{\sqrt{89}-5}$

Solution: (This question is MAML 1974, R5Q1.)



The actual location of the points isn't important, but notice that the closest point to $(-4,6)$ on the circle lies on the segment connecting $(-4,6)$ to the center of the circle. The distance to the center of the circle is $\sqrt{5^2+8^2}=\sqrt{89}$ and the radius of the circle is 5; therefore, the required distance to the circle is $\boxed{\sqrt{89}-5.}$

2. Find the value of k so that the graph of $x^2+2xy+y^2-2x-2y=k$ will be two parallel lines, one with y-intercept of 5.

Answer: $\boxed{15}$

Solution: (This question is MAML 1982, R5Q2.) With some factoring we get $(x+y)^2-2(x+y)-k=0$. This will factor into something of the form $((x+y)+a)((x+y)+b)=0$ (where $a+b=-2$ and $ab=-k$). Since $x+y+a=0$ or $x+y+b=0$, the solution set is indeed two lines.

If one of these lines (without loss of generality, the second) has a y-intercept of 5, then it passes through $(0,5)$. This means $0+5+b=0$, or $b=-5$.

Since $a+b=-2$, a must equal 3. So $-k=ab=(3)(-5)=-15$, whence $\boxed{k=15.}$

3. Determine the value of a so that the curve $y = x^2$ is tangent to the curve $y = a(x-3)^2 + 6$.

Answer: $\boxed{-2}$

Solution: (This question is MAML 1986 R5Q3.) The two curves intersect where

$$x^2 = a(x-3)^2 + 6$$

$$x^2 = a(x^2 - 6x + 9) + 6$$
$$x^2 = ax^2 - 6ax + 9a + 6$$
$$0 = (a-1)x^2 - 6ax + 9a + 6$$

Because there is a quadratic term, the number of solutions will depend on the discriminant: if it is positive, there will be two intersection points; if it is zero, one; if it is negative, none. For the two parabolas to be tangent, there must be one intersection point.

$$b^2 - 4ac$$
$$(-6a)^2 - 4(a-1)(9a+6)$$
$$36a^2 - 4(9a^2 - 3a - 6)$$
$$36a^2 - 36a^2 + 12a + 24$$
$$12a + 24$$

$12a + 24$ is equal to zero only when $\boxed{a = -2.}$

Round 6 — Trigonometry and Complex Numbers

1. Simplify as much as possible:

$$\frac{1-\cos\theta+\cos 2\theta}{\sin 2\theta-\sin\theta}$$

Answer: $\boxed{\cot\theta}$

Solution: (This question is MAML 1975, R6Q1.) Using the identities $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$, we get

$$\frac{1-\cos\theta+(2\cos^2\theta-1)}{(2\sin\theta\cos\theta)-\sin\theta}$$
$$\frac{2\cos^2\theta-\cos\theta}{2\sin\theta\cos\theta-\sin\theta}$$
$$\frac{\cos\theta(2\cos\theta-1)}{\sin\theta(2\cos\theta-1)},$$

which equals $\boxed{\cot\theta.}$

2. If $\tan 35^\circ = w$, express $\tan 10^\circ$ in terms of w.

Answer: $\boxed{\frac{1-w}{1+w}}$

Solution: (This question is MAML 1991, R6Q1.) Using the identity

$$\tan(a-b)=\frac{\tan a-\tan b}{1+\tan a\tan b}$$

we have

$$\tan(45^\circ-35^\circ)=\frac{\tan 45^\circ-\tan 35^\circ}{1+\tan 45^\circ\tan 35^\circ}.$$

Substituting $\tan 45^\circ = 1$ and $\tan 35^\circ = w$, we get

$$\tan(10^\circ)=\boxed{\frac{1-w}{1+w}.}$$

3. Let z denote a complex number and let $\overline{z}$ be its conjugate. Find all values of z for which $(z)(\overline{z}) = 5$ and $z^2+\overline{z^2}=6$.

Answer: $\boxed{2+i, 2-i, -2+i, -2-i \text{ (in any order)}}$

Solution: (This question is MAML 1979, T6.) Letting $z = a + bi$, we quickly get

$$(a+bi)(a-bi)=a^2+b^2=5$$

and

$$(a+bi)^2+(a-bi)^2=6$$
$$a^2-b^2+2abi+a^2-b^2-2abi=6$$
$$2(a^2-b^2)=6$$
$$a^2-b^2=3.$$

So $a^2-b^2=3$ and $a^2+b^2=5$, which quickly leads to $a^2=4$ and $b^2=1$, whence $a=\pm 2$ and $b=\pm 1$.

There are four complex numbers that fit these criteria: $\boxed{2+i, 2-i, -2+i, -2-i.}$

Team Round

1. The fraction 59/120 can be written as the sum of unit fractions in many ways. For example, $59/120 = 1/4 + 1/8 + 1/15 + 1/20$, where the sum of the denominators is 47. Determine the smallest possible sum of the denominators in such a sum of unit fractions.

   Answer: $\boxed{19}$

   Solution: (This question is MAML 1981, R1Q3.) The goal here is to break 59 into a sum of factors of 120, with the individual factors as large as possible so the unit fractions have small denominators. After writing out the available factors, we note that $59 = 24 + 20 + 15$, so

$$\frac{59}{120}$$
$$\frac{24}{120} + \frac{20}{120} + \frac{15}{120}$$
$$\frac{1}{5} + \frac{1}{6} + \frac{1}{8},$$

   and the required minimum sum is $5 + 6 + 8 = \boxed{19.}$

2. Find the ordered triple (x, y, z) of integers for which

$$\left(\frac{25}{8}\right)^x \left(\frac{6}{25}\right)^y \left(\frac{16}{15}\right)^z = 2.$$

   Answer: $\boxed{(3,2,2)}$

   Solution: (This question is MAML 1976, T4.) Factoring the given numbers into their consituent primes, we have

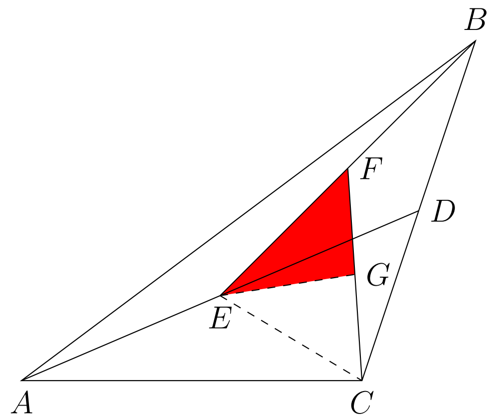
$$\left(\frac{5^2}{2^3}\right)^x \left(\frac{2\cdot 3}{5^2}\right)^y \left(\frac{2^4}{3\cdot 5}\right)^z = 2.$$
$$\frac{5^{2x}}{2^{3x}} \cdot \frac{2^y 3^y}{5^{2y}} \cdot \frac{2^{4z}}{3^z 5^z} = 2.$$
$$2^{y+4z-3x} \cdot 3^{y-z} \cdot 5^{2x-2y-z} = 2^1.$$

   This gives us a system of three equations: $y + 4z - 3x = 1$, $y - z = 0$, and $2x - 2y - z = 0$. The second implies that $y = z$, and plugging this into the first and third equations we have $5z - 3x = 1$ and $2x - 3z = 0$. The solution to this system is $x = 3$, $z = 2$, so the answer is $\boxed{(3,2,2).}$

3. In $\triangle ABC$, D is the midpoint of $\overline{BC}$, E is the midpoint of $\overline{AD}$, F is the midpoint of $\overline{BE}$, and G is the midpoint of $\overline{CF}$. Find the ratio of the area of $\triangle EFG$ to the area of $\triangle ABC$.

   Answer: $\boxed{1/8}$

   Solution: (This question is MAML 1978, R3Q3.)) Drawing a picture:



   Using the diagram, note that $[EFG] = \frac{1}{2}[EFC] = \frac{1}{4}[EBC] = \frac{1}{8}[ABC]$, so the answer is $\boxed{1/8.}$

4. Determine all real solutions to
$$x\sqrt{(x-1)^2} - 7x + 8 = 0.$$

   Answer: $\boxed{4+2\sqrt{2},\ 4-2\sqrt{2},\ -3-\sqrt{17} \text{ (in any order)}}$

   Solution: (This question is MAML 1985, R4Q3.) First, note that $\sqrt{a^2} \equiv |a|$. Therefore, the given equation becomes
$$x\,|x-1| - 7x + 8 = 0.$$
   Now, $x-1$ could be nonnegative or negative. If it is nonnegative, then $|x-1| = x-1$, and this equation becomes
$$x(x-1) - 7x + 8 = 0$$
$$x^2 - x - 7x + 8 = 0$$
$$x^2 - 8x + 8 = 0.$$
   The quadratic formula gives the solutions $\boxed{4 \pm 2\sqrt{2}}$, both of which are positive.

   If $x-1$ is negative, then $|x-1| = 1-x$, and this equation becomes
$$x(1-x) - 7x + 8 = 0$$
$$x - x^2 - 7x + 8 = 0$$
$$-x^2 - 6x + 8 = 0$$
$$x^2 + 6x - 8 = 0.$$
   The quadratic formula gives the solutions $-3 \pm \sqrt{17}$, but $\sqrt{17} > 4$, so $-3+\sqrt{17}$ is an extraneous solution. Therefore, the other solution, $\boxed{-3-\sqrt{17}}$, is the third and final solution.

5. The points of intersection of the graphs of

$$5x^2 - 3y^2 - 10x - 18y - 37 = 0$$

and

$$3x^2 - 5y^2 - 6x - 30y - 27 = 0$$

are the vertices of a convex quadrilateral. Find the area of this quadrilateral.

Answer: $\boxed{30}$

Solution: (This question is MAML 1979, T5.) The first equation can be transformed as follows:

$$5x^2 - 3y^2 - 10x - 18y - 37 = 0$$

$$\begin{aligned} 5x^2 - 10x - 3y^2 - 18y &= 37 \\ 5(x^2 - 2x) - 3(y^2 + 6y) &= 37 \\ 5(x^2 - 2x + 1) - 3(y^2 + 6y + 9) &= 37 + 5 - 27 \\ 5(x-1)^2 - 3(y+3)^2 &= 15 \end{aligned}$$

Similarly, the second equation can be transformed as follows:

$$3x^2 - 5y^2 - 6x - 30y - 27 = 0$$

$$\begin{aligned} 3x^2 - 6x - 5y^2 - 30y &= 27 \\ 3(x^2 - 2x) - 5(y^2 + 6y) &= 27 \\ 3(x^2 - 2x + 1) - 5(y^2 + 6y + 9) &= 27 + 3 - 45 \\ 3(x-1)^2 - 5(y+3)^2 &= -15 \end{aligned}$$

Both of these hyperbolas are centered at $(1, -3)$, which means we can translate both graphs to have their centers at $(0, 0)$ without affecting the area defined by their four intersection points. This gives us the system

$$\begin{aligned} 5u^2 - 3v^2 &= 15 \\ 3u^2 - 5v^2 &= -15 \end{aligned}$$

which has the solutions $u^2 = \frac{15}{2}$, $v^2 = \frac{15}{2}$. So the four intersection points are $\left(\pm\sqrt{\frac{15}{2}}, \pm\sqrt{\frac{15}{2}}\right)$, the vertices of a square.

Therefore the area of the square whose vertices are those four points is $\left(2\sqrt{\frac{15}{2}}\right)^2 = 4 \cdot \frac{15}{2} = \boxed{30.}$

6. Find the radian values of x, $0 \le x < 2\pi$, such that

$$\sin 3x - \sin 2x - \sin x = 0.$$

Answer: $\boxed{0, \frac{2\pi}{3}, \frac{4\pi}{3}, \pi \text{ (in any order)}}$

Solution: (This question is MAML 1985, R6Q3.) It is certainly possible to use (possibly derive) the formulas $\sin(2x) = 2\sin x \cos x$ and $\sin(3x) = 3\sin x - 4\sin^3 x$. But here's another way.

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$
$$\sin(a-b) = \sin a \cos b - \sin b \cos a$$

Subtracting, we get

$$\sin(a+b) - \sin(a-b) = 2\sin b \cos a.$$

By plugging in $a = 2x$ and $b = x$, we have

$$\sin 3x - \sin x = 2\sin x \cos 2x.$$

Therefore

$$\sin 3x - \sin 2x - \sin x = 2\sin x \cos 2x - 2\sin x \cos x.$$

When does this equal zero?

$$2\sin x \cos 2x - 2\sin x \cos x = 0$$
$$(2\sin x)(\cos 2x - \cos x) = 0$$
$$(2\sin x)((2\cos^2 x - 1) - \cos x) = 0$$
$$(2\sin x)(2\cos^2 x - \cos x - 1) = 0$$
$$(2\sin x)(2\cos x + 1)(\cos x - 1) = 0$$

In the interval $[0, 2\pi)$, $\sin x = 0$ at $\boxed{0 \text{ and } \pi}$; $\cos x = -\frac{1}{2}$ at $\boxed{\frac{2\pi}{3} \text{ and } \frac{4\pi}{3}}$; and $\cos x = 1$ at 0, which is a repeat.